

Computation of Minimal Filtered Free Resolutions over \mathbb{N} -Filtered Solvable Polynomial Algebras

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Abstract. Let $A = K[a_1, \dots, a_n]$ be a weighted \mathbb{N} -filtered solvable polynomial algebra with filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$, where solvable polynomial algebras are in the sense of (A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.*, 9(1990), 1–26), and FA is constructed with respect to a positive-degree function $d(\cdot)$ on A . By introducing minimal F-bases and minimal standard bases respectively for left A -modules and their submodules with respect to good filtrations, minimal filtered free resolutions for finitely generated A -modules are introduced. It is shown that any two minimal F-bases, respectively any two minimal standard bases have the same number of elements and the same number of elements of the same filtered degree; that minimal filtered free resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered A -modules; and that minimal finite filtered free resolutions can be algorithmically computed by employing Gröbner basis theory for modules over A with respect to any graded left monomial ordering on free left A -modules.

2010 Mathematics subject classification Primary 16W70; Secondary 16Z05.

Key words Solvable polynomial algebra, filtration, standard basis, Gröbner basis, free resolution.

1. Introduction and Preliminaries

In [Li3] it has been shown that the methods and algorithms, developed in ([CDNR], [KR]) for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over a commutative polynomial algebra, can be adapted for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over a weighted \mathbb{N} -graded (noncommutative) solvable polynomial algebra, and consequently, algorithmic procedures for computing minimal finite graded free resolutions over weighted \mathbb{N} -graded solvable

polynomial algebras are achieved, where a solvable polynomial algebra A is in the sense of [K-RW] and a weighted \mathbb{N} -gradation attached to A is specified to that constructed with respect to a positive-degree function $d(\cdot)$ on the PBW K -basis of A . Also we learnt from ([KR], Definition 4.2.13, Theorem 4.3.19) that if $L = \oplus_{i=1}^s Ae_i$ is a free module over a commutative polynomial algebra $A = K[x_1, \dots, x_n]$, then a minimal Macaulay basis for *any* finitely generated submodule N of L can be obtained via computing a minimal homogeneous generating set of the graded submodule generated by all leading homogeneous elements (degree forms) of N taking with respect to a fixed gradation for A and a fixed gradation for L , or can be obtained via computing a minimal homogeneous generating set of the graded submodule generated by all homogenized elements of N with respect to the variable x_0 in $\overline{A} = K[x_0, x_1, \dots, x_n]$ (though Theorem 4.3.19 in [KR] is mentioned only for ideals of A , this is true for submodules of L as we will see in later Section 5 where \overline{A} is replaced by the Rees algebra \tilde{A} of A). It is well known that Macaulay bases in more general context are called *standard bases*, especially standard bases for (two-sided) ideals in general *ungraded* commutative and noncommutative algebras were introduced by E.S. Golod ([Gol], 1986) in terms of the Γ -filtered structures attached to arbitrary associative algebras, where Γ is a well-ordered semigroup with respect to a well-ordering on Γ . Motivated by the work of [Gol], the results ([KR], Proposition 4.2.15, Proposition 4.3.21, Theorem 4.6.3, Proposition 4.7.24) and the work of [Li3], the aim of this paper is first to introduce minimal filtered free resolutions over a weighted \mathbb{N} -filtered solvable polynomial algebra A , and then, to give algorithmic procedures for computing minimal finite filtered free resolutions in the case where graded left monomial orderings on free left A -modules are used. Our goal will be reached by employing the filtered-graded transfer techniques and the Gröbner basis theory for solvable polynomial algebras as well as their modules. More clearly, the contents of this paper are arranged as follows:

1. Introduction and Preliminaries
2. Filtered Free Modules over Weighted \mathbb{N} -Filtered Solvable Polynomial Algebras
3. Filtered-Graded Transfer of Left Gröbner Bases for Modules
4. F-Bases and Standard Bases with Respect to Good Filtrations
5. Computation of Minimal F-Bases and Minimal Standard Bases
6. The Uniqueness of Minimal Filtered Free Resolutions
7. Computation of Minimal Finite Filtered Free Resolutions

Throughout this paper, K denotes a field, $K^* = K - \{0\}$; \mathbb{N} denotes the additive monoid of nonnegative integers, and \mathbb{Z} denotes the additive group of integers; all algebras are associative K -algebras with the multiplicative identity 1, and modules over an algebra are meant left unitary modules.

We start by recalling briefly some basics on Gröbner basis theory for solvable polynomial algebras and their modules. The main references are [AL1], [Gal], [K-RW], [Kr], [LW], [Li1], and [Lev]. Let $A = K[a_1, \dots, a_n]$ be a finitely generated K -algebra with the *minimal set of generators* $\{a_1, \dots, a_n\}$. If, for some permutation $\tau = i_1 i_2 \cdots i_n$ of $1, 2, \dots, n$, the set $\mathcal{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} \mid \alpha =$

$(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, forms a K -basis of A , then \mathcal{B} is referred to as a *PBW K -basis* of A . It is clear that if A has a PBW K -basis, then we can always assume that $i_1 = 1, \dots, i_n = n$. Thus, we make the following convention once for all.

Convention From now on in this paper, if we say that an algebra A has the PBW K -basis \mathcal{B} , then it means that

$$\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of \mathcal{B} are referred to as *monomials* of A .

Suppose that A has the PBW K -basis \mathcal{B} as presented above and that \prec is a total ordering on \mathcal{B} . Then every nonzero element $f \in A$ has a unique expression

$$f = \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \cdots + \lambda_m a^{\alpha(m)}, \quad \lambda_j \in K^*, \quad a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in \mathcal{B}, \quad 1 \leq j \leq m.$$

If $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)}$ in the above representation, then the *leading monomial* of f is defined as $\mathbf{LM}(f) = a^{\alpha(m)}$, the *leading coefficient* of F is defined as $\mathbf{LC}(f) = \lambda_m$, and the *leading term* of f is defined as $\mathbf{LT}(f) = \lambda_m a^{\alpha(m)}$.

1.1. Definition Suppose that the K -algebra $A = K[a_1, \dots, a_n]$ has the PBW K -basis \mathcal{B} . If \prec is a total ordering on \mathcal{B} that satisfies the following three conditions:

- (1) \prec is a well-ordering;
- (2) For $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\alpha \prec a^\beta$ and $\mathbf{LM}(a^\gamma a^\alpha a^\eta), \mathbf{LM}(a^\gamma a^\beta a^\eta) \notin K$, then $\mathbf{LM}(a^\gamma a^\alpha a^\eta) \prec \mathbf{LM}(a^\gamma a^\beta a^\eta)$;
- (3) For $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\beta \neq a^\gamma$, and $a^\gamma = \mathbf{LM}(a^\alpha a^\beta a^\eta)$, then $a^\beta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \neq 1$),

then \prec is called a *monomial ordering* on \mathcal{B} (or a monomial ordering on A).

If \prec is a monomial ordering on \mathcal{B} , then we call (\mathcal{B}, \prec) an *admissible system* of A .

Note that if a K -algebra $A = K[a_1, \dots, a_n]$ has the PBW K -basis $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, then for any given n -tuple $(m_1, \dots, m_n) \in \mathbb{N}^n$, a *weighted degree function* $d(\)$ is well defined on nonzero elements of A , namely, for each $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \in \mathcal{B}$, $d(a^\alpha) = m_1 \alpha_1 + \cdots + m_n \alpha_n$, and for each nonzero $f = \sum_{i=1}^s \lambda_i a^{\alpha(i)} \in A$ with $\lambda_i \in K^*$ and $a^{\alpha(i)} \in \mathcal{B}$, $d(f) = \max\{d(a^{\alpha(i)}) \mid 1 \leq i \leq s\}$. If $d(a_i) = m_i > 0$ for $1 \leq i \leq n$, then $d(\)$ is referred to as a *positive-degree function* on A .

Let $d(\)$ be a positive-degree function on A . If \prec is a monomial ordering on \mathcal{B} such that for all $a^\alpha, a^\beta \in \mathcal{B}$,

$$(*) \quad a^\alpha \prec a^\beta \text{ implies } d(a^\alpha) \leq d(a^\beta),$$

then we call \prec a *graded monomial ordering* with respect to $d(\)$, and from now on, unless otherwise stated we always use \prec_{gr} to denote a graded monomial ordering.

As one may see from the literature that in both the commutative and noncommutative computational algebra, the most popularly used graded monomial orderings on an algebra A with the PBW K -basis \mathcal{B} are those graded (reverse) lexicographic orderings with respect to the degree function $d(\cdot)$ such that $d(a_i) = 1$, $1 \leq i \leq n$.

1.2. Definition Suppose that the K -algebra $A = K[a_1, \dots, a_n]$ has an admissible system (\mathcal{B}, \prec) . If for all $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, $a^\beta = a_1^{\beta_1} \cdots a_n^{\beta_n} \in \mathcal{B}$, the following condition is satisfied:

$$\begin{aligned} a^\alpha a^\beta &= \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}, \\ \text{where } \lambda_{\alpha,\beta} &\in K^*, \quad a^{\alpha+\beta} = a_1^{\alpha_1+\beta_1} \cdots a_n^{\alpha_n+\beta_n}, \text{ and} \\ f_{\alpha,\beta} &\in K\text{-span}\mathcal{B} \text{ with } \mathbf{LM}(f_{\alpha,\beta}) \prec a^{\alpha+\beta} \text{ whenever } f_{\alpha,\beta} \neq 0, \end{aligned}$$

then A is said to be a *solvable polynomial algebra*.

Remark Let $A = K[a_1, \dots, a_n]$ be a finitely generated K -algebra and $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ the free K -algebra on $\{X_1, \dots, X_n\}$. Then it follows from [Li2] that A is a solvable polynomial algebra if and only if

- (1) $A \cong K\langle X \rangle / \langle G \rangle$ with a finite set of defining relations $G = \{g_1, \dots, g_m\}$ such that with respect to some monomial ordering \prec_x on $K\langle X \rangle$, G is a Gröbner basis of the ideal $\langle G \rangle$, and the set of normal monomials (mod G) gives rise to a PBW K -basis \mathcal{B} for A , and
- (2) there is a monomial ordering \prec on \mathcal{B} (not necessarily the restriction of \prec_x on \mathcal{B}) such that the condition on monomials given in Definition 1.2 is satisfied (see Example (3) given in the next section for an illustration).

Thus, solvable polynomial algebras are completely determinable and constructible in a computational way.

By Definition 1.2 it is straightforward that if A is a solvable polynomial algebra and $f, g \in A$ with $\mathbf{LM}(f) = a^\alpha$, $\mathbf{LM}(g) = a^\beta$, then

$$(\mathbb{P}1) \quad \mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g)) = \mathbf{LM}(a^\alpha a^\beta) = a^{\alpha+\beta}.$$

We shall freely use this property in the rest of this paper without additional indication.

The results mentioned in the Theorem below are summarized from ([K-RW], Sections 2 – 5).

1.3. Theorem Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with admissible system (\mathcal{B}, \prec) . The following statements hold.

- (i) A is a (left and right) Noetherian domain.
- (ii) With respect to the given \prec on \mathcal{B} , every nonzero left ideal I of A has a finite left Gröbner basis $\mathcal{G} = \{g_1, \dots, g_t\} \subset I$ in the sense that
 - if $f \in I$ and $f \neq 0$, then there is a $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i) | \mathbf{LM}(f)$, i.e., there is some $a^\gamma \in \mathcal{B}$ such that $\mathbf{LM}(f) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, with $\gamma(i_j) = (\gamma_{i_{1j}}, \gamma_{i_{2j}}, \dots, \gamma_{i_{nj}}) \in \mathbb{N}^n$, f has

a left Gröbner representation:

$$f = \sum_{i,j} \lambda_{ij} a^{\gamma(i_j)} g_j, \text{ where } \lambda_{ij} \in K^*, a^{\gamma(i_j)} \in \mathcal{B}, g_j \in \mathcal{G}, \\ \text{satisfying } \mathbf{LM}(a^{\gamma(i_j)} g_j) \preceq \mathbf{LM}(f) \text{ for all } (i, j).$$

(iii) The Buchberger algorithm, that computes a finite Gröbner basis for a finitely generated commutative polynomial ideal, has a complete noncommutative version that computes a finite left Gröbner basis for a finitely generated left ideal $I = \sum_{i=1}^m A f_i$ of A (see **Algorithm 1** given in the end of this section).

(iv) Similar results of (ii) and (iii) hold for right ideals and two-sided ideals of A .

□

Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with admissible system (\mathcal{B}, \prec) , and let $L = \bigoplus_{i=1}^s A e_i$ be a free (left) A -module with the A -basis $\{e_1, \dots, e_s\}$. Then L is a Noetherian module with the K -basis

$$\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}.$$

We also call elements of $\mathcal{B}(e)$ *monomials* in L . If \prec_e is a total ordering on $\mathcal{B}(e)$, and if $\xi = \sum_{j=1}^m \lambda_j a^{\alpha(j)} e_{i_j} \in L$, where $\lambda_j \in K^*$ and $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n}) \in \mathbb{N}^n$, such that $a^{\alpha(1)} e_{i_1} \prec_e a^{\alpha(2)} e_{i_2} \prec_e \dots \prec_e a^{\alpha(m)} e_{i_m}$, then by $\mathbf{LM}(\xi)$ we denote the *leading monomial* $a^{\alpha(m)} e_{i_m}$ of ξ , by $\mathbf{LC}(\xi)$ we denote the *leading coefficient* λ_m of ξ , and by $\mathbf{LT}(\xi)$ we denote the *leading term* $\lambda_m a^{\alpha(m)} e_{i_m}$ of f .

With respect to the given monomial ordering \prec on \mathcal{B} , a total ordering \prec_e on $\mathcal{B}(e)$ is called a *left monomial ordering* if the following two conditions are satisfied:

- (1) $a^\alpha e_i \prec_e a^\beta e_j$ implies $\mathbf{LM}(a^\gamma a^\alpha e_i) \prec_e \mathbf{LM}(a^\gamma a^\beta e_j)$ for all $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$, $a^\gamma \in \mathcal{B}$;
- (2) $a^\beta \prec a^\alpha$ implies $a^\alpha e_i \prec_e a^\beta e_i$ for all $a^\alpha, a^\beta \in \mathcal{B}$ and $1 \leq i \leq s$.

From the definition it is straightforward to check that every left monomial ordering \prec_e on $\mathcal{B}(e)$ is a well-ordering. Moreover, if $f \in A$ with $\mathbf{LM}(f) = a^\gamma$ and $\xi \in L$ with $\mathbf{LM}(\xi) = a^\alpha e_i$, then by referring to the foregoing (P1) we have

$$(\mathbb{P}2) \quad \mathbf{LM}(f\xi) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(\xi)) = \mathbf{LM}(a^\gamma a^\alpha e_i) = a^{\gamma+\alpha} e_i.$$

We shall also freely use this property in the rest of this paper without additional indication.

Actually as in the commutative case (cf. [AL2], [KR]), any left monomial ordering \prec on \mathcal{B} may induce two left monomial orderings on $\mathcal{B}(e)$:

$$\begin{aligned} (\mathbf{TOP} \text{ ordering}) \quad a^\alpha e_i \prec_e a^\beta e_j &\iff a^\alpha \prec a^\beta, \text{ or } a^\alpha = a^\beta \text{ and } i < j; \\ (\mathbf{POT} \text{ ordering}) \quad a^\alpha e_i \prec_e a^\beta e_j &\iff i < j, \text{ or } i = j \text{ and } a^\alpha \prec a^\beta. \end{aligned}$$

Let \prec_e be a left monomial ordering on the K -basis $\mathcal{B}(e)$ of L , and $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$, where $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. We say that $a^\alpha e_i$ *divides* $a^\beta e_j$, denoted $a^\alpha e_i | a^\beta e_j$, if $i = j$ and $a^\beta e_i = \mathbf{LM}(a^\gamma a^\alpha e_i)$ for some $a^\gamma \in \mathcal{B}$. It follows from the foregoing property (P2) that

$$a^\alpha e_i | a^\beta e_j \text{ if and only if } i = j \text{ and } \beta_i \geq \alpha_i, 1 \leq i \leq n.$$

This division of monomials can be extended to a division algorithm of dividing an element ξ by a finite subset of nonzero elements $U = \{\xi_1, \dots, \xi_m\}$ in L . That is, if there is some $\xi_{i_1} \in U$ such that $\mathbf{LM}(\xi_{i_1}) | \mathbf{LM}(\xi)$, i.e., there is a monomial $a^{\alpha(i_1)} \in \mathcal{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^{\alpha(i_1)}\xi_{i_1})$, then $\xi' := \xi - \frac{\mathbf{LC}(\xi)}{\mathbf{LC}(a^{\alpha(i_1)}\xi_{i_1})}a^{\alpha(i_1)}\xi_{i_1}$; otherwise, $\xi' := \xi - \mathbf{LT}(\xi)$. Executing this procedure for ξ' and so on, it follows from the well-ordering property of \prec_e that after finitely many repetitions ξ has an expression

$$\begin{aligned} \xi &= \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} \xi_j + \eta, \text{ where } \lambda_{ij} \in K, a^{\alpha(i_j)} \in \mathcal{B}, \xi_j \in U, \\ \eta &= 0 \text{ or } \eta = \sum_k \lambda_k a^{\gamma(k)} e_k \text{ with } \lambda_k \in K, a^{\gamma(k)} e_k \in \mathcal{B}(e), \\ &\text{satisfying} \\ \mathbf{LM}(a^{\alpha(i_j)} \xi_j) &\preceq_e \mathbf{LM}(\xi) \text{ for all } \lambda_{ij} \neq 0, \text{ and if } \eta \neq 0, \text{ then} \\ a^{\gamma(k)} e_k &\preceq_e \mathbf{LM}(\xi), \mathbf{LM}(\xi_i) \not\preceq_e a^{\gamma(k)} e_k \text{ for all } \xi_i \in U \text{ and all } \lambda_k \neq 0. \end{aligned}$$

The element η appeared in the above expression is called a *remainder* of ξ on division by U , and is usually denoted by $\bar{\xi}^U$, i.e., $\bar{\xi}^U = \eta$. If $\bar{\xi}^U = 0$, then we say that ξ is *reduced to zero* on division by U . A nonzero element $\xi \in L$ is said to be *normal* (mod U) if $\xi = \bar{\xi}^U$.

Based on the division algorithm, the notion of a *left Gröbner basis* for a submodule N of the free module $L = \bigoplus_{i=1}^s A e_i$ comes into play. Since A is a Noetherian domain, it follows that L is a Noetherian A -module and the following proposition holds.

1.4. Theorem With respect to the given \prec_e on $\mathcal{B}(e)$, every nonzero submodule N of L has a finite left Gröbner basis $\mathcal{G} = \{g_1, \dots, g_m\} \subset N$ in the sense that

- if $\xi \in N$ and $\xi \neq 0$, then $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$ for some $g_i \in \mathcal{G}$, i.e., there is a monomial $a^\gamma \in \mathcal{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, ξ has a *left Gröbner representation* $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} g_j$, where $\lambda_{ij} \in K^*$, $a^{\alpha(i_j)} \in \mathcal{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, $g_j \in \mathcal{G}$, satisfying $\mathbf{LM}(a^{\alpha(i_j)} g_j) \preceq_e \mathbf{LM}(\xi)$;

moreover, starting with any finite generating set of N , such a left Gröbner basis \mathcal{G} can be computed by running a noncommutative version of the commutative Buchberger algorithm (cf. [Bu1], [Bu2], [BW]) for modules over solvable polynomial algebras.

□

For the use of later Section 7, we recall the noncommutative version of the Buchberger algorithm for modules over solvable polynomial algebras as follows.

Let $N = \sum_{i=1}^m A \xi_i$ with $U = \{\xi_1, \dots, \xi_m\} \subset L$. For $\xi_i, \xi_j \in U$ with $1 \leq i < j \leq m$, $\mathbf{LM}(\xi_i) = a^{\alpha(i)} e_p$, $\mathbf{LM}(\xi_j) = a^{\alpha(j)} e_q$, where $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$, $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$, put $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$. The *left S-polynomial* of ξ_i and ξ_j is defined as

$$S_\ell(\xi_i, \xi_j) = \begin{cases} \frac{1}{\mathbf{LC}(a^{\gamma-\alpha(i)} \xi_i)} a^{\gamma-\alpha(i)} \xi_i - \frac{1}{\mathbf{LC}(a^{\gamma-\alpha(j)} \xi_j)} a^{\gamma-\alpha(j)} \xi_j, & \text{if } p = q \\ 0, & \text{if } p \neq q. \end{cases}$$

Algorithm 1

INPUT : $U = \{\xi_1, \dots, \xi_m\}$ OUTPUT : $\mathcal{G} = \{g_1, \dots, g_t\}$, a left Gröbner basis for $N = \sum_{i=1}^m A\xi_i$,INITIALIZATION : $m' := m$, $\mathcal{G} := \{g_1 = \xi_1, \dots, g_{m'} = \xi_m\}$,

$$\mathcal{S} := \left\{ S_\ell(g_i, g_j) \mid \begin{array}{l} g_i, g_j \in \mathcal{G}, i < j, \text{ and for some } e_t, \\ \mathbf{LM}(g_i) = a^\alpha e_t, \mathbf{LM}(g_j) = a^\beta e_t \end{array} \right\}$$

BEGIN

 WHILE $\mathcal{S} \neq \emptyset$ Choose any $S_\ell(g_i, g_j) \in \mathcal{S}$ $\mathcal{S} := \mathcal{S} - \{S_\ell(g_i, g_j)\}$ $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta$ IF $\eta \neq 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN $m' := m' + 1$, $g_{m'} := \eta$ $\mathcal{S} := \mathcal{S} \cup \{S_\ell(g_j, g_{m'}) \mid g_j \in \mathcal{G}, \mathbf{LM}(g_j) = a^\nu e_k\}$ $\mathcal{G} := \mathcal{G} \cup \{g_{m'}\}$,

END

END

END

One is referred to the up-to-date computer algebra system SINGULAR [DGPS] for a package implementing **Algorithm 1**.

2. Filtered Free Modules over Weighted \mathbb{N} -Filtered Solvable Polynomial Algebras

Recall that the \mathbb{N} -filtered solvable polynomial algebras with the generators of degree 1 (especially the quadric solvable polynomial algebras) were studied in ([LW], [Li1]). In this section, by means of positive-degree functions we introduce more generally the weighted \mathbb{N} -filtered solvable polynomial algebras and filtered free left modules over such algebras. Since the standard bases we are going to introduce in terms of good filtrations are generalization of classical Macaulay bases (see a remark given in Section 4), while a classical Macaulay basis V is characterized in terms of both the leading homogeneous elements (degree forms) of V and the homogenized elements of V (cf. [KR], P.38, P.55), accordingly both the associated graded algebra (module) and the Rees algebra (module) of an \mathbb{N} -filtered algebra (module) come into play in our noncommutative filtered context. All notions, notations and conventions used in Section 1 are maintained.

Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with admissible system (\mathcal{B}, \prec) , where $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is the PBW K -basis of A and \prec is a monomial ordering on \mathcal{B} , and let $d(\)$ be a positive-degree function on A such that $d(a_i) = m_i > 0$, $1 \leq i \leq n$

(see Section 2). Put

$$F_p A = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) \leq p\}, \quad p \in \mathbb{N},$$

then it is clear that $F_p A \subseteq F_{p+1} A$ for all $p \in \mathbb{N}$, $A = \cup_{p \in \mathbb{N}} F_p A$, and $1 \in F_0 A = K$.

2.1. Definition With notation as above, if $F_p A F_q A \subseteq F_{p+q} A$ holds for all $p, q \in \mathbb{N}$, then we call A a weighted \mathbb{N} -filtered solvable polynomial algebra with respect to the given positive-degree function $d(\cdot)$ on A , and accordingly we call $FA = \{F_p A\}_{p \in \mathbb{N}}$ the weighted \mathbb{N} -filtration of A .

Note that a weighted \mathbb{N} -filtration FA of A is clearly *separated* in the sense that if f is a nonzero element of A , then $f \in F_p A - F_{p-1} A$ for some p . Thus, if $f \in F_p A - F_{p-1} A$, then we say that f has *filtered degree* p and we use $d_{\text{fil}}(f)$ to denote this degree, i.e.,

$$(\mathbb{P}3) \quad d_{\text{fil}}(f) = p \iff f \in F_p A - F_{p-1} A.$$

Since our definition of a weighted \mathbb{N} -filtration FA for A depends on a positive-degree function $d(\cdot)$ on A , bearing in mind $(\mathbb{P}3)$ above and the following featured property of FA will very much help us to deal with the associated graded structures of A and filtered A -modules.

2.2. Lemma If $f = \sum_i \lambda_i a^{\alpha(i)}$ with $\lambda_i \in K^*$ and $a^{\alpha(i)} \in \mathcal{B}$, then $d_{\text{fil}}(f) = p$ if and only if $d(a^{\alpha(i')}) = p$ for some i' if and only if $d(f) = p = d_{\text{fil}}(f)$. □

Given a solvable polynomial algebra $A = K[a_1, \dots, a_n]$ with admissible system (\mathcal{B}, \prec) , it follows from Definition 1.2, Definition 2.1 and Lemma 2.2 that

- A is a weighted \mathbb{N} -filtered solvable polynomial algebra with respect to a given positive-degree function $d(\cdot)$ if and only if for $1 \leq i < j \leq n$, in the relation $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ with $f_{ji} = \sum \mu_k a^{\alpha(k)}$, $d(a^{\alpha(k)}) \leq d(a_i a_j)$ holds whenever $\mu_k \neq 0$.

This observation helps us to better understand the following examples.

Example (1) If $A = K[a_1, \dots, a_n]$ is a weighted \mathbb{N} -graded solvable polynomial algebra with respect to a positive-degree function $d(\cdot)$ on A , i.e., $A = \oplus_{p \in \mathbb{N}} A_p$ with each $A_p = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) = p\}$ the degree- p homogeneous part (see [Li3] for more details on weighted \mathbb{N} -graded solvable polynomial algebras), then, with respect to the same positive-degree function $d(\cdot)$ on A , A is turned into a weighted \mathbb{N} -filtered solvable polynomial algebra with the \mathbb{N} -filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ where each $F_p A = \oplus_{q \leq p} A_q$.

Example (2) Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathcal{B}, \prec_{gr})$, where \prec_{gr} is a graded monomial ordering on \mathcal{B} with respect to a given positive-degree function $d(\cdot)$ on A (see the definition of \prec_{gr} given in Section 1). Then by referring to Definition 1.2 and the above observation, one easily sees that A is a weighted \mathbb{N} -filtered solvable polynomial algebra

with respect to the same $d(\cdot)$. In the case where \prec_{gr} respects $d(a_i) = 1$ for $1 \leq i \leq n$, Definition 1.2 entails that the generators of A satisfy

$$a_j a_i = \lambda_{ji} a_i a_j + \sum \lambda_{k\ell}^{ji} a_k a_\ell + \sum \lambda_t^{ji} a_t + \mu_{ji}, \quad 1 \leq i < j \leq n, \quad \lambda_{ji} \in K^*, \quad \lambda_{k\ell}^{ji}, \lambda_t^{ji}, \mu_{ji} \in K.$$

In [Li1] such \mathbb{N} -filtered solvable polynomial algebras are referred to as *quadric solvable polynomial algebras* which include numerous significant algebras such as Weyl algebras and enveloping algebras of Lie algebras. One is referred to [Li1] for some detailed study of quadric solvable polynomial algebras by means of the filtered-graded transfer of Gröbner bases.

The next example provides weighted \mathbb{N} -filtered solvable polynomial algebras in which some generators may have degree ≥ 2 .

Example (3) Considering the \mathbb{N} -graded structure of the free K -algebra $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ by assigning X_1 the degree 2, X_2 the degree 1 and X_3 the degree 4, let I be the ideal of $K\langle X \rangle$ generated by the elements

$$\begin{aligned} g_1 &= X_1 X_2 - X_2 X_1, \\ g_2 &= X_3 X_1 - \lambda X_1 X_3 - \mu X_3 X_2^2 - f(X_2), \\ g_3 &= X_3 X_2 - X_2 X_3, \end{aligned}$$

where $\lambda \in K^*$, $\mu \in K$, and $f(X_2) \in K\text{-span}\{1, X_2, X_2^2, \dots, X_2^6\}$. If we use the graded lexicographic ordering $X_2 \prec_{grlex} X_1 \prec_{grlex} X_3$ on $K\langle X \rangle$, then it is straightforward to verify that $\mathcal{G} = \{g_1, g_2, g_3\}$ forms a Gröbner basis for I , and that $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$ is a PBW basis for the K -algebra A , where $A = K[a_1, a_2, a_3] = K\langle X \rangle / I$ with a_1, a_2 and a_3 denoting the cosets $X_1 + I$, $X_2 + I$ and $X_3 + I$ in $K\langle X \rangle / I$ respectively. Since $a_3 a_1 = \lambda a_1 a_3 + \mu a_2^2 a_3 + f(a_2)$, where $f(a_2) \in K\text{-span}\{1, a_2, a_2^2, \dots, a_2^6\}$, we see that A has the monomial ordering \prec_{lex} on \mathcal{B} such that $a_3 \prec_{lex} a_2 \prec_{lex} a_1$ and $\mathbf{LM}(\mu a_2^2 a_3 + f(a_2)) \prec_{lex} a_1 a_3$, thereby A is turned into a weighted \mathbb{N} -filtered solvable polynomial algebra with respect to \prec_{lex} and the degree function $d(\cdot)$ such that $d(a_1) = 2$, $d(a_2) = 1$, and $d(a_3) = 4$. Moreover, one may also check that with respect to the same degree function $d(\cdot)$, the graded lexicographic ordering $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$ is another choice to make A into a weighted \mathbb{N} -filtered solvable polynomial algebra.

Let A be a weighted \mathbb{N} -filtered solvable polynomial algebra with admissible system (\mathcal{B}, \prec) , and let $FA = \{F_p A\}_{p \in \mathbb{N}}$ be the \mathbb{N} -filtration of A constructed with respect to a given positive-degree function $d(\cdot)$ on A . Then A has the associated \mathbb{N} -graded K -algebra $G(A) = \bigoplus_{p \in \mathbb{N}} G(A)_p$ with $G(A)_0 = F_0 A = K$ and $G(A)_p = F_p A / F_{p-1} A$ for $p \geq 1$, where for $\bar{f} = f + F_{p-1} A \in G(A)_p$, $\bar{g} = g + F_{q-1} A$, the multiplication is given by $\bar{f}\bar{g} = fg + F_{p+q-1} A \in G(A)_{p+q}$. Another \mathbb{N} -graded K -algebra determined by FA is the Rees algebra \tilde{A} of A , which is defined as $\tilde{A} = \bigoplus_{p \in \mathbb{N}} \tilde{A}_p$ with $\tilde{A}_p = F_p A$, where the multiplication of \tilde{A} is induced by $F_p A F_q A \subseteq F_{p+q} A$, $p, q \in \mathbb{N}$. For convenience, we fix the following notations once for all:

- If $h \in G(A)_p$ and $h \neq 0$, then we write $d_{gr}(h)$ for the degree of h as a homogeneous element of $G(A)$, i.e., $d_{gr}(h) = p$.

- If $H \in \tilde{A}_p$ and $H \neq 0$, then we write $d_{\text{gr}}(H)$ for the degree of H as a homogeneous element of \tilde{A} , i.e., $d_{\text{gr}}(H) = p$.

Concerning the \mathbb{N} -graded structure of $G(A)$, if $f \in A$ with $d_{\text{fil}}(f) = p$, then by Lemma 2.2, the coset $f + F_{p-1}A$ represented by f in $G(A)_p$ is a nonzero homogeneous element of degree p . If we denote this homogeneous element by $\sigma(f)$ (in the literature it is referred to as the principal symbol of f), then $d_{\text{fil}}(f) = p = d_{\text{gr}}(\sigma(f))$. However, considering the Rees algebra \tilde{A} of A , we note that a nonzero $f \in F_q A$ represents a homogeneous element of degree q in \tilde{A}_q on one hand, and on the other hand it represents a homogeneous element of degree q_1 in \tilde{A}_{q_1} , where $q_1 = d_{\text{fil}}(f) \leq q$. So, for a nonzero $f \in F_p A$, we denote the corresponding homogeneous element of degree p in \tilde{A}_p by $h_p(f)$, while we use \tilde{f} to denote the homogeneous element represented by f in \tilde{A}_{p_1} with $p_1 = d_{\text{fil}}(f) \leq p$. Thus, $d_{\text{gr}}(\tilde{f}) = d_{\text{fil}}(f)$, and we see that $h_p(f) = \tilde{f}$ if and only if $d_{\text{fil}}(f) = p$.

Furthermore, if we write Z for the homogeneous element of degree 1 in \tilde{A}_1 represented by the multiplicative identity element 1, then Z is a central regular element of \tilde{A} , i.e., Z is not a divisor of zero and is contained in the center of \tilde{A} . Bringing this homogeneous element Z into play, the homogeneous elements of \tilde{A} are featured as follows:

- If $f \in A$ with $d_{\text{fil}}(f) = p_1$ then for all $p \geq p_1$, $h_p(f) = Z^{p-p_1} \tilde{f}$. In other words, if $H \in \tilde{A}_p$ is any nonzero homogeneous element of degree p , then there is some $f \in F_p A$ such that $H = Z^{p-d(f)} \tilde{f} = \tilde{f} + (Z^{p-d(f)} - 1) \tilde{f}$.

It follows that by sending H to $f + F_{p-1}A$ and sending H to f respectively, $G(A) \cong \tilde{A}/\langle Z \rangle$ as \mathbb{N} -graded K -algebras and $A \cong \tilde{A}/\langle 1 - Z \rangle$ as K -algebras (cf. [AVV], [LVO]).

Since a solvable polynomial algebra A is necessarily a domain, we summarize two useful properties concerning the multiplication of $G(A)$ and \tilde{A} respectively into the following lemma. Notations are as given before.

2.3. Lemma Let f, g be nonzero elements of A with $d_{\text{fil}}(f) = p_1$, $d_{\text{fil}}(g) = p_2$. Then

- (i) $\sigma(f)\sigma(g) = \sigma(fg)$;
- (ii) $\tilde{f}\tilde{g} = \tilde{fg}$. If $p_1 + p_2 \leq p$, then $h_p(fg) = Z^{p-p_1-p_2} \tilde{f}\tilde{g}$.

□

With the preparation made above, the results given in the next theorem, which are analogues of those concerning quadric solvable polynomial algebras in ([LW], Section 3; [Li1], CH.IV), may be derived in a similar way as in loc. cit., thereby detailed proofs are omitted.

2.4. Theorem Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathcal{B}, \prec_{gr})$, where \prec_{gr} is a graded monomial ordering on \mathcal{B} with respect to a given positive-degree function $d(\cdot)$ on A , thereby A is a weighted \mathbb{N} -filtered solvable polynomial algebra with respect to the same $d(\cdot)$ by the foregoing Example (2), and let $FA = \{F_p A\}_{p \in \mathbb{N}}$ be the corresponding \mathbb{N} -filtration of A . Considering the associated graded algebra $G(A)$ as well as the Rees algebra \tilde{A} of A , the following

statements hold.

(i) $G(A) = K[\sigma(a_1), \dots, \sigma(a_n)]$, $G(A)$ has the PBW K -basis

$$\sigma(\mathcal{B}) = \{\sigma(a)^\alpha = \sigma(a_1)^{\alpha_1} \cdots \sigma(a_n)^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$

and, by referring to Definition 1.2, for $\sigma(a)^\alpha, \sigma(a)^\beta \in \sigma(\mathcal{B})$ such that $a^\alpha a^\beta = \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}$, where $\lambda_{\alpha,\beta} \in K^*$, if $f_{\alpha,\beta} = 0$ then

$$\sigma(a)^\alpha \sigma(a)^\beta = \lambda_{\alpha,\beta} \sigma(a)^{\alpha+\beta}, \text{ where } \sigma(a)^{\alpha+\beta} = \sigma(a_1)^{\alpha_1+\beta_1} \cdots \sigma(a_n)^{\alpha_n+\beta_n};$$

and in the case where $f_{\alpha,\beta} = \sum_j \mu_j^{\alpha,\beta} a^{\alpha(j)} \neq 0$ with $\mu_j^{\alpha,\beta} \in K$,

$$\sigma(a)^\alpha \sigma(a)^\beta = \lambda_{\alpha,\beta} \sigma(a)^{\alpha+\beta} + \sum_{d(a^{\alpha(k)})=d(a^{\alpha+\beta})} \mu_j^{\alpha,\beta} \sigma(a)^{\alpha(k)}.$$

Moreover, the ordering $\prec_{G(A)}$ defined on $\sigma(\mathcal{B})$ subject to the rule:

$$\sigma(a)^\alpha \prec_{G(A)} \sigma(a)^\beta \iff a^\alpha \prec_{gr} a^\beta, \quad a^\alpha, a^\beta \in \mathcal{B},$$

is a graded monomial ordering with respect to the positive-degree function $d(\cdot)$ on $G(A)$ such that $d(\sigma(a_i)) = d(a_i)$ for $1 \leq i \leq n$, that turns $G(A)$ into a weighted \mathbb{N} -graded solvable polynomial algebra.

(ii) $\tilde{A} = K[\tilde{a}_1, \dots, \tilde{a}_n, Z]$ where Z is the central regular element of degree 1 in \tilde{A}_1 represented by 1, \tilde{A} has the PBW K -basis

$$\tilde{\mathcal{B}} = \{\tilde{a}^\alpha Z^m = \tilde{a}_1^{\alpha_1} \cdots \tilde{a}_n^{\alpha_n} Z^m \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, m \in \mathbb{N}\},$$

and, by referring to Definition 1.2, for $\sigma(a)^\alpha, \sigma(a)^\beta \in \sigma(\mathcal{B})$ such that $a^\alpha a^\beta = \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}$, where $\lambda_{\alpha,\beta} \in K^*$, if $f_{\alpha,\beta} = 0$ then

$$\tilde{a}^\alpha Z^s \cdot \tilde{a}^\beta Z^t = \lambda_{\alpha,\beta} \tilde{a}^{\alpha+\beta} Z^{s+t}, \text{ where } \tilde{a}^{\alpha+\beta} = \tilde{a}_1^{\alpha_1+\beta_1} \cdots \tilde{a}_n^{\alpha_n+\beta_n};$$

and in the case where $f_{\alpha,\beta} = \sum_j \mu_j^{\alpha,\beta} a^{\alpha(j)} \neq 0$ with $\mu_j^{\alpha,\beta} \in K$,

$$\tilde{a}^\alpha Z^s \cdot \tilde{a}^\beta Z^t = \lambda_{\alpha,\beta} \tilde{a}^{\alpha+\beta} Z^{s+t} + \sum_j \mu_j^{\alpha,\beta} \tilde{a}^{\alpha(j)} Z^{q-m_j}, \text{ where } q = d(a^{\alpha+\beta}) + s + t, m_j = d(a^{\alpha(j)}).$$

Moreover, the ordering $\prec_{\tilde{A}}$ defined on $\tilde{\mathcal{B}}$ subject to the rule:

$$\tilde{a}^\alpha Z^s \prec_{\tilde{A}} \tilde{a}^\beta Z^t \iff a^\alpha \prec_{gr} a^\beta, \text{ or } a^\alpha = a^\beta \text{ and } s < t, \quad a^\alpha, a^\beta \in \mathcal{B},$$

is a monomial ordering on $\tilde{\mathcal{B}}$ (which is not necessarily a graded monomial ordering), that turns \tilde{A} into a weighted \mathbb{N} -graded solvable polynomial algebra with respect to the positive-degree function $d(\cdot)$ on \tilde{A} such that $d(Z) = 1$ and $d(\tilde{a}_i) = d(a_i)$ for $1 \leq i \leq n$.

□

By referring to Lemma 2.2 and Lemma 2.3, the corollary presented below is straightforward and will be very often used in discussing left Gröbner bases and standard bases in free left A -modules and their associated graded free $G(A)$ -modules as well the graded free \tilde{A} -modules (Section 3, Section 4).

2.5. Corollary With the assumption and notations as in Theorem 2.4, if $f = \lambda a^\alpha + \sum_j \mu_j a^{\alpha(j)}$ with $d(f) = p$ and $\mathbf{LM}(f) = a^\alpha$, then $p = d_{\text{fil}}(f) = d_{\text{gr}}(\sigma(f)) = d_{\text{gr}}(\tilde{f})$, and

$$\begin{aligned}\sigma(f) &= \lambda \sigma(a)^\alpha + \sum_{d(a^{\alpha(j_k)})=p} \mu_{j_k} \sigma(a)^{\alpha(j_k)}; \\ \mathbf{LM}(\sigma(f)) &= \sigma(a)^\alpha = \sigma(\mathbf{LM}(f)); \\ \tilde{f} &= \lambda \tilde{a}^\alpha + \sum_j \mu_j \tilde{a}^{\alpha(j)} Z^{p-d(a^{\alpha(j)})}; \\ \mathbf{LM}(\tilde{f}) &= \tilde{a}^\alpha = \widetilde{\mathbf{LM}(f)},\end{aligned}$$

where $\mathbf{LM}(f)$, $\mathbf{LM}(\sigma(f))$ and $\mathbf{LM}(\tilde{f})$ are taken with respect to \prec_{gr} , $\prec_{G(A)}$ and $\prec_{\tilde{A}}$ respectively. \square

Let A be a weighted \mathbb{N} -filtered solvable polynomial algebra with admissible system (\mathcal{B}, \prec) , and let $FA = \{F_p A\}_{p \in \mathbb{N}}$ be the \mathbb{N} -filtration of A constructed with respect to a given positive-degree function $d(\cdot)$ on A (see Section 2). Consider a free A -module $L = \bigoplus_{i=1}^s A e_i$ with the A -basis $\{e_1, \dots, e_s\}$. Then L has the K -basis $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$. If $\{b_1, \dots, b_s\}$ is an *arbitrarily* fixed subset of \mathbb{N} , then, with $FL = \{F_q L\}_{q \in \mathbb{N}}$ defined by putting

$$F_q L = \{0\} \text{ if } q < \min\{b_1, \dots, b_s\}; \text{ otherwise } F_q L = \sum_{i=1}^s \left(\sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i,$$

or alternatively, for $q \geq \min\{b_1, \dots, b_s\}$,

$$F_q L = K\text{-span}\{a^\alpha e_i \in \mathcal{B}(e) \mid d(a^\alpha) + b_i \leq q\},$$

L forms an \mathbb{N} -filtered free A -module with respect to the \mathbb{N} -filtered structure of A , that is, every $F_q L$ is a K -subspace of L , $F_q L \subseteq F_{q+1} L$ for all $q \in \mathbb{N}$, $L = \bigcup_{q \in \mathbb{N}} F_q L$, $F_p A F_q L \subseteq F_{p+q} L$ for all $p, q \in \mathbb{N}$, and for each $i = 1, \dots, s$,

$$e_i \in F_0 L \text{ if } b_i = 0; \text{ otherwise } e_i \in F_{b_i} L - F_{b_i-1} L.$$

Convention Let A be a weighted \mathbb{N} -filtered solvable polynomial algebra. Unless otherwise stated, from now on in the subsequent sections if we say that $L = \bigoplus_{i=1}^s A e_i$ is a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$, then FL is always meant the type as constructed above.

Let $L = \bigoplus_{i=1}^s A e_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$, which is constructed with respect to a given subset $\{b_1, \dots, b_s\} \subset \mathbb{N}$. Then FL is *separated* in the sense that if ξ is a nonzero element of L , then $\xi \in F_q L - F_{q-1} L$ for some q . Thus, to make the discussion on

FL compatible with FA , if $\xi \in F_q L - F_{q-1} L$, then we say that ξ has *filtered degree* q and we use $d_{\text{fil}}(\xi)$ to denote this degree, i.e.,

$$(\mathbb{P}4) \quad d_{\text{fil}}(\xi) = q \iff \xi \in F_q L - F_{q-1} L.$$

For instance, we have $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$. Comparing with Lemma 2.2 we first note the following

2.6. Lemma Let $\xi \in L$. Then $d_{\text{fil}}(\xi) = q$ if and only if $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$, where $\lambda_{ij} \in K^*$ and $a^{\alpha(i_j)} \in \mathcal{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, in which some monomial $a^{\alpha(i_j)} e_j$ satisfy $d(a^{\alpha(i_j)}) + b_j = q$. □

Let $L = \bigoplus_{i=1}^s A e_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$. Considering the the associated \mathbb{N} -graded algebra $G(A)$ of A , the filtered free A module L has the *associated \mathbb{N} -graded $G(A)$ -module* $G(L) = \bigoplus_{q \in \mathbb{N}} G(L)_q$ with $G(L)_q = F_q L / F_{q-1} L$, where for $\bar{f} = f + F_{p-1} A \in G(A)_p$, $\bar{\xi} = \xi + F_{q-1} L \in G(L)_q$, the module action is given by $\bar{f} \cdot \bar{\xi} = f\xi + F_{p+q-1} L \in G(L)_{p+q}$. As with homogeneous elements in $G(A)$, if $h \in G(L)_q$ and $h \neq 0$, then we write $d_{\text{gr}}(h)$ for the degree of h as a homogeneous element of $G(L)$, i.e., $d_{\text{gr}}(h) = q$. If $\xi \in L$ with $d_{\text{fil}}(\xi) = q$, then the coset $\xi + F_{q-1} L$ represented by ξ in $G(L)_q$ is a nonzero homogeneous element of degree q , and if we denote this homogeneous element by $\sigma(\xi)$ (in the literature it is referred to as the principal symbol of ξ) then $d_{\text{gr}}(\sigma(\xi)) = q = d_{\text{fil}}(\xi)$.

Furthermore, considering the Rees algebra \tilde{A} of A , the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ of L also defines the *Rees module* \tilde{L} of L , which is the \mathbb{N} -graded \tilde{A} -module $\tilde{L} = \bigoplus_{q \in \mathbb{N}} \tilde{L}_q$, where $\tilde{L}_q = F_q L$ and the module action is induced by $F_p A F_q L \subseteq F_{p+q} L$. As with homogeneous elements in \tilde{A} , if $H \in \tilde{L}_q$ and $H \neq 0$, then we write $d_{\text{gr}}(H)$ for the degree of H as a homogeneous element of \tilde{L} , i.e., $d_{\text{gr}}(H) = q$. Note that any nonzero $\xi \in F_q L$ represents a homogeneous element of degree q in \tilde{L}_q on one hand, and on the other hand it represents a homogeneous element of degree q_1 in \tilde{L}_{q_1} , where $q_1 = d_{\text{fil}}(\xi) \leq q$. So, for a nonzero $\xi \in F_q L$ we denote the corresponding homogeneous element of degree q in \tilde{L}_q by $h_q(\xi)$, while we use $\tilde{\xi}$ to denote the homogeneous element represented by ξ in \tilde{L}_{q_1} with $q_1 = d_{\text{fil}}(\xi) \leq q$. Thus, $d_{\text{gr}}(\tilde{\xi}) = d_{\text{fil}}(\xi)$, and we see that $h_q(\xi) = \tilde{\xi}$ if and only if $d_{\text{fil}}(\xi) = q$.

We also note that if Z denotes the homogeneous element of degree 1 in \tilde{A}_1 represented by the multiplicative identity element 1, then, similar to the discussion given before Theorem 2.4, there are A -module isomorphism $L \cong \tilde{L}/(1 - Z)\tilde{L}$ and graded $G(A)$ -module isomorphism $G(L) \cong \tilde{L}/Z\tilde{L}$ (cf. [LVVO], [LVO]).

2.7. Lemma With notation as above, the following statements hold.

- (i) $d_{\text{fil}}(f\xi) = d(f) + d_{\text{fil}}(\xi)$ holds for all nonzero $f \in A$ and nonzero $\xi \in L$.
- (ii) $\sigma(f)\sigma(\xi) = \sigma(f\xi)$ holds for all nonzero $f \in A$ and nonzero $\xi \in L$.
- (iii) If $\xi \in L$ with $d_{\text{fil}}(\xi) = q \leq \ell$, then $h_\ell(\xi) = Z^{\ell-q} \tilde{\xi}$. Furthermore, let $f \in A$ with $d_{\text{fil}}(f) = p$, $\xi \in L$ with $d_{\text{fil}}(\xi) = q$. Then $\widetilde{f\xi} = \widetilde{f}\tilde{\xi}$; if $p + q \leq \ell$, then $h_\ell(f\xi) = Z^{\ell-p-q} \widetilde{f}\tilde{\xi}$.

Proof Since A is a solvable polynomial algebra, $G(A)$ and \tilde{A} are \mathbb{N} -graded solvable polynomial algebras by Theorem 2.4, thereby they are necessarily domains. By the foregoing (P3), (P4) and Lemma 2.6, the verification of (i) – (iii) are then straightforward. \square

2.8. Proposition With notation as fixed before, let $L = \bigoplus_{i=1}^s A e_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$. The following two statements hold.

(i) $G(L)$ is an \mathbb{N} -graded free $G(A)$ -module with the homogeneous $G(A)$ -basis $\{\sigma(e_1), \dots, \sigma(e_s)\}$, that is, $G(L) = \bigoplus_{i=1}^s G(A)\sigma(e_i) = \bigoplus_{q \in \mathbb{N}} G(L)_q$ with

$$G(L)_q = \sum_{p_i + b_i = q} G(A)_{p_i} \sigma(e_i) \quad q \in \mathbb{N}.$$

Moreover, $\sigma(\mathcal{B}(e)) = \{\sigma(a^\alpha e_i) = \sigma(a)^\alpha \sigma(e_i) \mid a^\alpha e_i \in \mathcal{B}(e)\}$ forms a K -basis for $G(L)$.

(ii) \tilde{L} is an \mathbb{N} -graded free \tilde{A} -module with the homogeneous \tilde{A} -basis $\{\tilde{e}_1, \dots, \tilde{e}_s\}$, that is, $\tilde{L} = \bigoplus_{i=1}^s \tilde{A} \tilde{e}_i = \bigoplus_{q \in \mathbb{N}} \tilde{L}_q$ with

$$\tilde{L}_q = \sum_{p_i + b_i = q} \tilde{A}_{p_i} \tilde{e}_i, \quad q \in \mathbb{N}.$$

Moreover, $\widetilde{\mathcal{B}(e)} = \{\tilde{a}^\alpha Z^m \tilde{e}_i \mid \tilde{a}^\alpha Z^m \in \tilde{\mathcal{B}}, 1 \leq i \leq s\}$ forms a K -basis for \tilde{L} , where $\tilde{\mathcal{B}}$ is the PBW K -basis of \tilde{A} determined in Theorem 4.4(ii).

Proof Since $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, if $\xi = \sum_{i=1}^s f_i e_i \in F_q L = \sum_{i=1}^s \left(\sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i$, then $d_{\text{fil}}(\xi) \leq q$. By Lemma 2.7,

$$\sigma(\xi) = \sum_{d(f_i) + b_i = q} \sigma(f_i) \sigma(e_i) \in \sum_{i=1}^s G(A)_{q-b_i} \sigma(e_i)$$

$$h_q(\xi) = \sum_{i=1}^s Z^{q-d(f_i)-b_i} \tilde{f}_i \tilde{e}_i \in \sum_{i=1}^s \tilde{A}_{q-b_i} \tilde{e}_i.$$

This shows that $\{\sigma(e_1), \dots, \sigma(e_s)\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_s\}$ generate the $G(A)$ -module $G(L)$ and the \tilde{A} -module \tilde{L} , respectively. Next, since each $\sigma(e_i)$ is a homogeneous element of degree b_i , if a degree- q homogeneous element $\sum_{i=1}^s \sigma(f_i) \sigma(e_i) = 0$, where $f_i \in A$, $d_{\text{fil}}(f_i) + b_i = q$, $1 \leq i \leq s$, then $\sum_{i=1}^s f_i e_i \in F_{q-1} L$ and hence each $f_i \in F_{q-1-b_i} A$ by Lemma 2.6, a contradiction. It follows that $\{\sigma(e_1), \dots, \sigma(e_s)\}$ is linearly independent over $G(A)$. Concerning the linear independence of $\{\tilde{e}_1, \dots, \tilde{e}_s\}$ over \tilde{A} , since each \tilde{e}_i is a homogeneous element of degree b_i , if a degree- q homogeneous element $\sum_{i=1}^s h_{p_i}(f_i) \tilde{e}_i = 0$, where $f_i \in F_{p_i} A$ and $p_i + b_i = q$, $1 \leq i \leq s$, then $\sum_{i=1}^s f_i e_i = 0$ in $F_q L$ and consequently all $f_i = 0$, thereby $h_{p_i}(f_i) = 0$ as desired. Finally, if $\xi \in F_q L$ with $d_{\text{fil}}(\xi) = q$, then by Lemma 2.6, $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$ with $\lambda_{ij} \in K^*$ and $d(a^{\alpha(i_j)}) + b_j = \ell_{ij} \leq q$. It follows from Lemma 2.7 that

$$\begin{aligned} \sigma(\xi) &= \sum_{\ell_{ik}=q} \lambda_{ik} \sigma(a)^{\alpha(i_k)} \sigma(e_k), \\ \tilde{\xi} &= \sum_{i,j} \lambda_{ij} Z^{q-\ell_{ij}} \tilde{a}^{\alpha(i_j)} \tilde{e}_j. \end{aligned}$$

Therefore, a further application of Lemma 2.6 and Lemma 2.7 shows that $\sigma(\mathcal{B}(e))$ and $\widetilde{\mathcal{B}(e)}$ are K -bases for $G(L)$ and \tilde{L} respectively.

3. Filtered-Graded Transfer of Left Gröbner Bases for Modules

Throughout this section, we let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathcal{B}, \prec_{gr})$, where $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is the PBW K -basis of A and \prec_{gr} is a graded monomial ordering with respect to some given positive-degree function $d(\cdot)$ on A (see Section 2). Thereby A is turned into a weighted \mathbb{N} -filtered solvable polynomial algebra with the filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ constructed with respect to the same $d(\cdot)$ (see Example (2) of Section 2). In order to compute minimal standard bases by employing both inhomogeneous and homogenous left Gröbner bases in later Section 5, our aim of the current section is to show the relations between left Gröbner bases in a filtered free (left) A -module L and homogeneous left Gröbner bases in $G(L)$ as well as homogeneous left Gröbner bases in \tilde{L} , which are just module theory analogues of the results on filtered-graded transfer of Gröbner bases given in ([LW], [Li1]). All notions, notations and conventions introduced in previous sections are maintained.

Let $L = \oplus_{i=1}^s A e_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$. Bearing in mind Lemma 2.6, we say that a left monomial ordering on $\mathcal{B}(e)$ is a *graded left monomial ordering*, denoted by \prec_{e-gr} , if for $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$,

$$a^\alpha e_i \prec_{e-gr} a^\beta e_j \text{ implies } d_{\text{fil}}(a^\alpha e_i) = d(a^\alpha) + b_i \leq d(a^\beta) + b_j = d_{\text{fil}}(a^\beta e_j).$$

For instance, with respect to the given graded monomial ordering \prec_{gr} on \mathcal{B} and the \mathbb{N} -filtration FA of A , if $\{f_1, \dots, f_s\} \subset A$ is a finite subset such that $d(f_i) = b_i = d_{\text{fil}}(e_i)$, $1 \leq i \leq s$, then, by mimicking the Schreyer ordering in the commutative case (see [Sch], or [AL2], P.166), one may directly check that the ordering \prec_{s-gr} on $\mathcal{B}(e)$ induced by $\{f_1, \dots, f_s\}$ defined subject to the rule: for $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$,

$$a^\alpha e_i \prec_{s-gr} a^\beta e_j \iff \begin{cases} \mathbf{LM}(a^\alpha f_i) \prec_{gr} \mathbf{LM}(a^\beta f_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha f_i) = \mathbf{LM}(a^\beta f_j) \text{ and } i < j. \end{cases}$$

is a graded left monomial ordering on $\mathcal{B}(e)$.

More generally, let $\{\xi_1, \dots, \xi_m\} \subset L$ be a finite subset, where $d_{\text{fil}}(\xi_i) = q_i$, $1 \leq i \leq m$, and let $L_1 = \oplus_{i=1}^m A \xi_i$ be the filtered free A -module with the filtration $FL_1 = \{F_q L_1\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(\xi_i) = q_i$, $1 \leq i \leq m$. Then, given *any* graded left monomial ordering \prec_{e-gr} on $\mathcal{B}(e)$, the Schreyer ordering \prec_{s-gr} defined on the K -basis $\mathcal{B}(\varepsilon) = \{a^\alpha \varepsilon_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq m\}$ of L_1 subject to the rule: for $a^\alpha \varepsilon_i, a^\beta \varepsilon_j \in \mathcal{B}(\varepsilon)$,

$$a^\alpha \varepsilon_i \prec_{s-gr} a^\beta \varepsilon_j \iff \begin{cases} \mathbf{LM}(a^\alpha \xi_i) \prec_{e-gr} \mathbf{LM}(a^\beta \xi_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha \xi_i) = \mathbf{LM}(a^\beta \xi_j) \text{ and } i < j, \end{cases}$$

is a graded left monomial ordering on $\mathcal{B}(\varepsilon)$.

Comparing with Lemma 2.2 and Lemma 2.6, the lemma given below reveals the intrinsic property of a graded left monomial ordering employed by a filtered free A -module.

3.1. Lemma Let $L = \bigoplus_{i=1}^s Ae_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, and let \prec_{e-gr} be a graded left monomial ordering on $\mathcal{B}(e)$. Then \prec_{e-gr} is compatible with the filtration FL of L in the sense that $\xi \in F_q L - F_{q-1} L$, i.e. $d_{\text{fil}}(\xi) = q$, if and only if $\mathbf{LM}(\xi) = a^\alpha e_i$ with $d_{\text{fil}}(a^\alpha e_i) = d(a^\alpha) + b_i = q$.

Proof Let $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(ij)} e_j \in F_q L - F_{q-1} L$. Then by Lemma 2.6, there is some $a^{\alpha(i_\ell)} e_\ell$ such that $d(a^{\alpha(i_\ell)}) + b_\ell = q$. If $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$ with respect to \prec_{e-gr} , then $a^{\alpha(i_k)} e_k \prec_{e-gr} a^{\alpha(i_t)} e_t$ for all $a^{\alpha(i_k)} e_k$ with $k \neq t$. If $\ell = t$, then $d(a^{\alpha(i_t)}) + b_t = q$; otherwise, since \prec_{e-gr} is a graded left monomial ordering, we have $d(a^{\alpha(i_k)}) + b_k \leq d(a^{\alpha(i_t)}) + b_t$, in particular, $q = d(a^{\alpha(i_\ell)}) + b_\ell \leq d(a^{\alpha(i_t)}) + b_t \leq q$. Hence $d_{\text{fil}}(a^{\alpha(i_t)} e_t) = d(a^{\alpha(i_t)}) + b_t = q$, as desired.

Conversely, for $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(ij)} e_j \in L$, if, with respect to \prec_{e-gr} , $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$ with $d_{\text{fil}}(a^{\alpha(i_t)} e_t) = d(a^{\alpha(i_t)}) + b_t = q$, then $a^{\alpha(i_k)} e_k \prec_{e-gr} a^{\alpha(i_t)} e_t$ for all $k \neq t$. Since \prec_{e-gr} is a graded left monomial ordering, we have $d(a^{\alpha(i_k)}) + b_k \leq d(a^{\alpha(i_t)}) + b_t = q$. It follows from Lemma 2.6 that $d_{\text{fil}}(\xi) = q$, i.e., $\xi \in F_q L - F_{q-1} L$. \square

Let $L = \bigoplus_{i=1}^s Ae_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$. Then, by Proposition 2.8 we know that the associated graded $G(A)$ -module $G(L)$ of L is an \mathbb{N} -graded free module, i.e., $G(L) = \bigoplus_{i=1}^s G(A) \sigma(e_i)$ with the homogeneous $G(A)$ -basis $\{\sigma(e_1), \dots, \sigma(e_s)\}$, and that $G(L)$ has the K -basis $\sigma(\mathcal{B}(e)) = \{\sigma(a^\alpha e_i) = \sigma(a)^\alpha \sigma(e_i) \mid a^\alpha e_i \in \mathcal{B}(e)\}$. Furthermore, let \prec_{e-gr} be a graded left monomial ordering on $\mathcal{B}(e)$ as defined in the beginning of this section. Then we may define an ordering $\prec_{\sigma(e)-gr}$ on $\sigma(\mathcal{B}(e))$ subject to the rule:

$$\sigma(a)^\alpha \sigma(e_i) \prec_{\sigma(e)-gr} \sigma(a)^\beta \sigma(e_j) \iff a^\alpha e_i \prec_{e-gr} a^\beta e_j, \quad a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e).$$

3.2. Lemma With the ordering $\prec_{\sigma(e)-gr}$ defined above, the following statements hold.

- (i) $\prec_{\sigma(e)-gr}$ is a graded left monomial ordering on $\sigma(\mathcal{B}(e))$.
- (ii) (Compare with Corollary 2.5.) $\mathbf{LM}(\sigma(\xi)) = \sigma(\mathbf{LM}(\xi))$ holds for all nonzero $\xi \in L$, where the monomial orderings used for $\mathbf{LM}(\sigma(\xi))$ and $\mathbf{LM}(\xi)$ are $\prec_{\sigma(e)-gr}$ and \prec_{e-gr} respectively.

Proof (i) Noticing that the given monomial ordering \prec_{gr} on A is a graded monomial ordering with respect to a positive-degree function $d(\cdot)$ on A , it follows from Theorem 2.4(i) that $G(A)$ is turned into an \mathbb{N} -graded solvable polynomial algebra by using the graded monomial ordering $\prec_{G(A)}$ defined on $\sigma(\mathcal{B})$ subject to the rule: $\sigma(a)^\alpha \prec_{G(A)} \sigma(a)^\beta \iff a^\alpha \prec_{gr} a^\beta$, where the positive-degree function on $G(A)$ is given by $d(\sigma(a_i)) = d(a_i)$, $1 \leq i \leq n$. Moreover, since $\sigma(e_i)$ is a homogeneous element of degree b_i in $G(L)$, $1 \leq i \leq s$, by Lemma 2.7, it is then straightforward to verify that $\prec_{\sigma(e)-gr}$ is a graded left monomial ordering on $\sigma(\mathcal{B}(e))$.

(ii) Let $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(ij)} e_j$, where $\lambda_{ij} \in K^*$ and $a^{\alpha(ij)} \in \mathcal{B}$ with $\alpha(ij) = (\alpha_{i j_1}, \dots, \alpha_{i j_n}) \in \mathbb{N}^n$. If $d_{\text{fil}}(\xi) = q$, i.e., $\xi \in F_q L - F_{q-1} L$, then by Lemma 3.1, $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$ for some t such that

$d_{\text{fil}}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$. Since \prec_{e-gr} is a left graded monomial ordering on $\mathcal{B}(e)$, by Lemma 2.7 we have $\sigma(\xi) = \lambda_{it}\sigma(a)^{\alpha(i_t)}\sigma(e_t) + \sum_{d(a^{\alpha(i_k)})+b_k=q} \lambda_{ik}\sigma(a)^{\alpha(i_k)}\sigma(e_k)$. It follows from the definition of $\prec_{\sigma(e)-gr}$ that $\mathbf{LM}(\sigma(\xi)) = \sigma(a)^{\alpha(i_t)}\sigma(e_t) = \sigma(\mathbf{LM}(\xi))$, as desired. \square

3.3. Theorem Let N be a submodule of the filtered free A -module $L = \bigoplus_{i=1}^s Ae_i$, where L is equipped with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, and let \prec_{e-gr} be a graded left monomial ordering on $\mathcal{B}(e)$. For a subset $\mathcal{G} = \{g_1, \dots, g_m\}$ of N , the following two statements are equivalent.

- (i) \mathcal{G} is a left Gröbner basis of N with respect to \prec_{e-gr} .
- (ii) Putting $\sigma(\mathcal{G}) = \{\sigma(g_1), \dots, \sigma(g_m)\}$ and considering the filtration $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$ of N induced by FL (see Section 2), $\sigma(\mathcal{G})$ is a left Gröbner basis for the associated graded module $G(N)$ of N with respect to the graded left monomial ordering $\prec_{\sigma(e)-gr}$ defined above.

Proof (i) \Rightarrow (ii) Note that any nonzero homogeneous element of $G(N)$ is of the form $\sigma(\xi)$ with $\xi \in N$. If \mathcal{G} is a left Gröbner basis of N , then there exists some $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$, i.e., there is a monomial $a^\alpha \in \mathcal{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$. Since the given left monomial ordering \prec_{e-gr} on $\mathcal{B}(e)$ is a graded left monomial ordering, it follows from Lemma 2.7 and Lemma 3.2 that

$$\begin{aligned} \mathbf{LM}(\sigma(\xi)) &= \sigma(\mathbf{LM}(\xi)) \\ &= \sigma(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))) \\ &= \mathbf{LM}(\sigma(a^\alpha \mathbf{LM}(g_i))) \\ &= \mathbf{LM}(\sigma(a)^\alpha \sigma(\mathbf{LM}(g_i))) \\ &= \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i))). \end{aligned}$$

This shows that $\mathbf{LM}(\sigma(g_i)) | \mathbf{LM}(\sigma(\xi))$, thereby $\sigma(\mathcal{G})$ is a left Gröbner basis for $G(N)$.

(ii) \Rightarrow (i) Suppose that $\sigma(\mathcal{G})$ is a left Gröbner basis of $G(N)$ with respect to $\prec_{\sigma(e)-gr}$. If $\xi \in N$ and $\xi \neq 0$, then $\sigma(\xi) \neq 0$, and there exists a $\sigma(g_i) \in \sigma(\mathcal{G})$ such that $\mathbf{LM}(\sigma(g_i)) | \mathbf{LM}(\sigma(\xi))$, i.e., there is a monomial $\sigma(a)^\alpha \in \sigma(\mathcal{B})$ such that $\mathbf{LM}(\sigma(\xi)) = \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i)))$. Again as \prec_{e-gr} is a left graded monomial ordering on $\mathcal{B}(e)$, by Lemma 2.7 and Lemma 3.2 we have

$$\begin{aligned} \sigma(\mathbf{LM}(\xi)) &= \mathbf{LM}(\sigma(\xi)) \\ &= \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i))) \\ &= \mathbf{LM}(\sigma(a)^\alpha \sigma(\mathbf{LM}(g_i))) \\ &= \mathbf{LM}(\sigma(a^\alpha \mathbf{LM}(g_i))) \\ &= \sigma(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))). \end{aligned}$$

This shows that $d_{\text{fil}}(\mathbf{LM}(\xi)) = d_{\text{fil}}(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i)))$. Since both $\mathbf{LM}(\xi)$ and $\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$ are monomials in $\mathcal{B}(e)$, it follows from the construction of FL and Lemma 3.1 that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$, i.e., $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$. This shows that \mathcal{G} is a left Gröbner basis for N . \square

Similarly, in light of Proposition 2.8 we may define an ordering $\prec_{\tilde{e}}$ on the K -basis $\widetilde{\mathcal{B}(e)} = \{Z^m \tilde{a}^\alpha \tilde{e}_i \mid Z^m \tilde{a}^\alpha \in \tilde{\mathcal{B}}, 1 \leq i \leq s\}$ of the \mathbb{N} -graded free \tilde{A} -module $\tilde{L} = \bigoplus_{i=1}^s \tilde{A} \tilde{e}_i$ subject to the rule:

for $Z^s \tilde{a}^\alpha \tilde{e}_i, Z^t \tilde{a}^\beta \tilde{e}_j \in \widetilde{\mathcal{B}(e)}$,

$$Z^s \tilde{a}^\alpha \tilde{e}_i \prec_{\tilde{e}} Z^t \tilde{a}^\beta \tilde{e}_j \iff a^\alpha e_i \prec_{e-gr} a^\beta e_j, \text{ or } a^\alpha e_i = a^\beta e_j \text{ and } s < t.$$

3.4. Lemma With the ordering $\prec_{\tilde{e}}$ defined above, the following statements hold.

- (i) $\prec_{\tilde{e}}$ is a left monomial ordering on $\widetilde{\mathcal{B}(e)}$.
- (ii) (Compare with Corollary 2.5.) $\mathbf{LM}(\tilde{\xi}) = \widetilde{\mathbf{LM}(\xi)}$ holds for all nonzero $\xi \in L$, where the monomial orderings used for $\mathbf{LM}(\tilde{\xi})$ and $\mathbf{LM}(\xi)$ are $\prec_{\tilde{e}}$ and \prec_{e-gr} respectively.

Proof (i) Noticing that the given monomial ordering \prec_{gr} for A is a graded monomial ordering with respect to a positive-degree function $d(\cdot)$ on A , it follows from Theorem 2.4(ii) that \tilde{A} is turned into an $\mathbb{N}_{\tilde{\mathcal{B}}}$ -graded solvable polynomial algebra by using the monomial ordering $\prec_{\tilde{A}}$ defined on $\tilde{\mathcal{B}}$ subject to the rule: $\tilde{a}^\alpha Z^s \prec_{\tilde{A}} \tilde{a}^\beta Z^t \iff a^\alpha \prec_{gr} a^\beta$, or $a^\alpha = a^\beta$ and $s < t$, $a^\alpha, a^\beta \in \mathcal{B}$, where the positive-degree function on \tilde{A} is given by $d(\tilde{a}_i) = d(a_i)$ for $1 \leq i \leq n$, and $d(Z) = 1$. Moreover, since \tilde{e}_i is a homogeneous element of degree b_i in \tilde{A} , $1 \leq i \leq s$, by Lemma 2.7, it is then straightforward to verify that $\prec_{\tilde{e}}$ is a left monomial ordering on $\widetilde{\mathcal{B}(e)}$.

(ii) Let $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$, where $\lambda_{ij} \in K^*$ and $a^{\alpha(i_j)} \in \mathcal{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$. If $d_{\text{fil}}(\xi) = q$, i.e., $\xi \in F_q L - F_{q-1} L$, then by Lemma 3.1, $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$ for some t such that $d_{\text{fil}}(a^{\alpha(i_t)} e_t) = d(a^{\alpha(i_t)}) + b_t = q$. Since \prec_{e-gr} is a left graded monomial ordering on $\mathcal{B}(e)$, by Lemma 2.7 we have $\tilde{\xi} = \lambda_{it} \tilde{a}^{\alpha(i_t)} \tilde{e}_t + \sum_{j \neq t} \lambda_{ij} Z^{q - \ell_{ij}} \tilde{a}^{\alpha(i_j)} \tilde{e}_j$, where $\ell_{ij} = d_{\text{fil}}(a^{\alpha(i_j)} e_j) = d(a^{\alpha(i_j)}) + d_j$. It follows from the definition of $\prec_{\tilde{e}}$ that $\mathbf{LM}(\tilde{\xi}) = \tilde{a}^{\alpha(i_t)} \tilde{e}_t = \widetilde{\mathbf{LM}(\xi)}$, as desired. \square

3.5. Theorem Let N be a submodule of the filtered free A -module $L = \bigoplus_{i=1}^s A e_i$, where L is equipped with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, and let \prec_{e-gr} be a graded left monomial ordering on $\mathcal{B}(e)$. For a subset $\mathcal{G} = \{g_1, \dots, g_m\}$ of N , the following two statements are equivalent.

- (i) \mathcal{G} is a left Gröbner basis of N with respect to \prec_{e-gr} .
- (ii) Putting $\tau(\mathcal{G}) = \{\tilde{g}_1, \dots, \tilde{g}_m\}$ and considering the filtration $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$ of N induced by FL (see Section 2), $\tau(\mathcal{G})$ is a left Gröbner basis for the Rees module \tilde{N} of N with respect to the left monomial ordering $\prec_{\tilde{e}}$ defined above.

Proof (i) \Rightarrow (ii) Note that any nonzero homogeneous element of \tilde{N} is of the form $h_q(\xi)$ for some $\xi \in F_q N$ with $d_{\text{fil}}(\xi) = q_1 \leq q$. By Lemma 2.7, $h_q(\xi) = Z^{q-q_1} \tilde{\xi}$. If \mathcal{G} is a left Gröbner basis of N , then there exists some $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$, i.e., there is a monomial $a^\alpha \in \mathcal{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$. It follows from Lemma 2.7 and Lemma 3.4 that

$$\begin{aligned} \mathbf{LM}(\tilde{\xi}) &= \widetilde{\mathbf{LM}(\xi)} \\ &= (\mathbf{LM}(a^\alpha \mathbf{LM}(g_i)))^\sim \\ &= \mathbf{LM}((a^\alpha \mathbf{LM}(g_i))^\sim) \\ &= \mathbf{LM}(\tilde{a}^\alpha \mathbf{LM}(g_i)) \\ &= \mathbf{LM}(\tilde{a}^\alpha \mathbf{LM}(\tilde{g}_i)). \end{aligned}$$

Hence, noticing the definition of $\prec_{\tilde{e}}$ we have

$$\begin{aligned}\mathbf{LM}(h_q(\xi)) &= \mathbf{LM}(Z^{q-q_1}\tilde{\xi}) \\ &= Z^{q-q_1}\mathbf{LM}(\tilde{\xi}) \\ &= Z^{q-q_1}\mathbf{LM}(\tilde{a}^\alpha\mathbf{LM}(\tilde{g}_i)) \\ &= \mathbf{LM}(z^{q-q_1}\tilde{a}^\alpha\mathbf{LM}(\tilde{g}_i)).\end{aligned}$$

This shows that $\mathbf{LM}(\tilde{g}_i)|\mathbf{LM}(h_q(\xi))$, thereby $\tau(\mathcal{G})$ is a left Gröbner basis of \tilde{N} .

(ii) \Rightarrow (i) If $\xi \in N$ and $\xi \neq 0$, then $\tilde{\xi} \neq 0$ and $\mathbf{LM}(\tilde{\xi}) = \widetilde{\mathbf{LM}(\xi)}$ by Lemma 3.4. Suppose that $\tau(\mathcal{G})$ is a left Gröbner basis of \tilde{N} with respect to $\prec_{\tilde{e}}$. Then there exists some $\tilde{g}_i \in \tau(\mathcal{G})$ such that $\mathbf{LM}(\tilde{g}_i)|\mathbf{LM}(\tilde{\xi})$, i.e., there is a monomial $Z^m\tilde{a}^\gamma \in \tilde{\mathcal{B}}$ such that $\mathbf{LM}(\tilde{\xi}) = \mathbf{LM}(Z^m\tilde{a}^\gamma\mathbf{LM}(\tilde{g}_i))$. Since the given left monomial ordering \prec_{e-gr} on $\mathcal{B}(e)$ is a graded left monomial ordering, it follows from Lemma 2.7, the definition of $\prec_{\tilde{e}}$ and Lemma 3.2 that

$$\begin{aligned}\widetilde{a^\alpha e_j} = \widetilde{\mathbf{LM}(\xi)} = \mathbf{LM}(\tilde{\xi}) &= \mathbf{LM}(Z^m\tilde{a}^\gamma\mathbf{LM}(\tilde{g}_i)) \\ &= Z^m(\mathbf{LM}((a^\gamma\mathbf{LM}(g_i))^\sim)) \\ &= Z^m(\mathbf{LM}(a^\gamma\mathbf{LM}(g_i)))^\sim.\end{aligned}$$

Noticing the discussion on \tilde{L} and the role played by Z given before Lemma 2.7, we must have $m = 0$, thereby $\mathbf{LM}(\xi) = \mathbf{LM}(a^\gamma\mathbf{LM}(g_i))$. This shows that \mathcal{G} is a left Gröbner basis for N .

Remark It is known that Gröbner bases for ungraded ideals in both a commutative polynomial algebra and a noncommutative free algebra can be obtained via computing homogeneous Gröbner bases for graded ideals in the corresponding homogenized (graded) algebras (cf. [Fröb], [LS], [Li4]). Similarly for a weighted \mathbb{N} -filtered solvable polynomial algebra A , by using a (de)homogenization-like trick with respect to the central regular element Z in \tilde{A} , the discussion on \tilde{A} and \tilde{L} presented in Section 2 indeed enables us to obtain left Gröbner bases of submodules (left ideals) in L (in A) via computing homogeneous left Gröbner bases of graded submodules (graded left ideals) in \tilde{L} (in \tilde{A}). Since this topic is beyond the scope of this paper, we omit the detailed discussion here.

4. F-Bases and Standard Bases with Respect to Good Filtrations

Let $A = K[a_1, \dots, a_n]$ be a weighted \mathbb{N} -filtered solvable polynomial algebra with admissible system (\mathcal{B}, \prec) and the \mathbb{N} -filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ constructed with respect to a given positive-degree function $d(\cdot)$ on A (see Section 2). In this section, we introduce F-bases and standard bases respectively for \mathbb{N} -filtered left A -modules and their submodules with respect to good filtrations, and we show that any two minimal F-bases, respectively any two minimal standard bases have the same number of elements and the same number of elements of the same filtered degree. Moreover, we show that a standard basis for a submodule N of a filtered free A -module L can be obtained via computing a

left Gröbner basis of N with respect to a graded left monomial ordering. All notions, notations and conventions used before are maintained.

Let M be an A -module. Recall that M is said to be an \mathbb{N} -filtered A -module if M has a filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, where each $F_q M$ is a K -subspace of M , such that $M = \cup_{q \in \mathbb{N}} F_q M$, $F_q M \subseteq F_{q+1} M$ for all $q \in \mathbb{N}$, and $F_p A F_q M \subseteq F_{p+q} M$ for all $p, q \in \mathbb{N}$.

Convention Unless otherwise stated, from now on in the subsequent sections a filtered A -module M is always meant an \mathbb{N} -filtered module with a filtration of the type $FM = \{F_q M\}_{q \in \mathbb{N}}$ as described above.

Let $G(A)$ be the associated graded algebra of A , \tilde{A} the Rees algebra of A , and Z the homogeneous element of degree 1 in \tilde{A}_1 represented by the multiplicative identity 1 of A (see Section 2). If M is a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, then M has the associated graded $G(A)$ -module $G(M) = \oplus_{q \in \mathbb{N}} G(M)_q$ with $G(M)_0 = F_0 M$ and $G(M)_q = F_q M / F_{q-1} M$ for $q \geq 1$, and the Rees module of M is defined as the graded \tilde{A} -module $\tilde{M} = \oplus_{q \in \mathbb{N}} \tilde{M}_q$ with each $\tilde{M}_q = F_q M$. As with a filtered free A -module in Section 2, we have $\tilde{M}/Z\tilde{M} \cong G(M)$ as graded $G(A)$ -modules, and $\tilde{M}/(1-Z)\tilde{M} \cong M$ as A -modules. Moreover, we may also define the filtered degree of a nonzero $\xi \in M$, that is, $d_{\text{fil}}(\xi) = q$ if and only if $\xi \in F_q M - F_{q-1} M$. So, actually as in Section 2, for $\xi \in M$ with $d_{\text{fil}}(\xi) = q$, if we write $\sigma(\xi)$ for the nonzero homogeneous element of degree q represented by ξ in $G(M)_q$, $\tilde{\xi}$ for the degree- q homogeneous element represented by ξ in \tilde{M}_q , and $h_{q'}(\xi)$ for the degree- q' homogeneous element represented by ξ in $\tilde{M}_{q'}$ with $q < q'$, then $d_{\text{fil}}(\xi) = q = d_{\text{gr}}(\sigma(\xi)) = d_{\text{gr}}(\tilde{\xi})$, and $d_{\text{gr}}(h_{q'}(\xi)) = q'$.

With notation as fixed above, the lemma presented below is a version of ([LVO], Ch.I, Lemma 5.4, Theorem 5.7) for \mathbb{N} -filtered modules.

4.1. Lemma Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and $V = \{v_1, \dots, v_m\}$ a finite subset of nonzero elements in M . The following statements are equivalent:

(i) There is a subset $S = \{n_1, \dots, n_m\} \subset \mathbb{N}$ such that

$$F_q M = \sum_{i=1}^m \left(\sum_{p_i + n_i \leq q} F_{p_i} A \right) v_i, \quad q \in \mathbb{N};$$

(ii) $G(M) = \sum_{i=1}^m G(A) \sigma(v_i)$;

(iii) $\tilde{M} = \sum_{i=1}^m \tilde{A} \tilde{v}_i$.

□

4.2. Definition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and let $V = \{v_1, \dots, v_m\} \subset M$ be a finite subset of nonzero elements. If V satisfies one of the equivalent conditions of Lemma 4.1, then we call V an F -basis of M with respect to FM .

Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$. If V is an F-basis of M with respect to FM , then it is necessary to note that

- (1) since $M = \cup_{q \in \mathbb{N}} F_q M$, it is clear that V is certainly a generating set of the A -module M , i.e., $M = \sum_{i=1}^m A v_i$;
- (2) due to Lemma 4.1(i), the filtration FM is usually referred to as a *good filtration* of M in the literature concerning filtered module theory (cf. [LVO]).

Indeed, if an A -module $M = \sum_{i=1}^t A u_i$ is finitely generated by the subset $U = \{u_1, \dots, u_t\}$, and if $S = \{n_1, \dots, n_t\}$ is an arbitrarily chosen subset of \mathbb{N} , then U is an F-basis of M with respect to the good filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$ defined by setting

$$F_q M = \{0\} \text{ if } q < \min\{n_1, \dots, n_m\}; \text{ otherwise } F_q M = \sum_{i=1}^t \left(\sum_{p_i + n_i \leq q} F_{p_i} A \right) u_i, \quad q \in \mathbb{N}.$$

In particular, if $L = \oplus_{i=1}^s A e_i$ is a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ as constructed in Section 2 such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, then $\{e_1, \dots, e_s\}$ is an F-basis of L with respect to the good filtration FL .

4.3. Definition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and suppose that M has an F-basis $V = \{v_1, \dots, v_m\}$ with respect to FM . If any proper subset of V cannot be an F-basis of M with respect to FM , then we say that V is a *minimal F-basis* of M with respect to FM .

Note that A is a weighted \mathbb{N} -filtered K -algebra such that $G(A) = \oplus_{p \in \mathbb{N}} G(A)_p$ with $G(A)_0 = K$, $\tilde{A} = \oplus_{p \in \mathbb{N}} \tilde{A}_p$ with $\tilde{A}_0 = K$, while K is a field. by Lemma 4.1 and the classical result on graded modules over an \mathbb{N} -graded algebra with the degree-0 homogeneous part a field, we have immediately the following

4.4. Proposition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and $V = \{v_1, \dots, v_m\} \subset M$ a subset of nonzero elements. Then V is a minimal F-basis of M with respect to FM if and only if $\sigma(V) = \{\sigma(v_1), \dots, \sigma(v_m)\}$ is a minimal homogeneous generating set of $G(M)$ if and only if $\tau(V) = \{\tilde{v}_1, \dots, \tilde{v}_m\}$ is a minimal homogeneous generating set of \tilde{M} . Hence, any two minimal F-bases of M with respect to FM have the same number of elements and the same number of elements of the same filtered degree.

□

Let M be an \mathbb{N} -filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and let N be a submodule of M with the filtration $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ induced by FM . Then, as with a filtered free A -module in Section 2, the associated graded $G(A)$ -module $G(N) = \oplus_{q \in \mathbb{N}} G(N)_q$ of N with $G(N)_q = F_q N / F_{q-1} N$ is a graded submodule of $G(M)$, and the Rees module $\tilde{N} = \oplus_{q \in \mathbb{N}} \tilde{N}_q$ of N with $\tilde{N}_q = F_q N$ is a graded submodule of \tilde{M} .

4.5. Definition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and let N be a submodule of M . Consider the filtration $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ of N induced by FM . If $W = \{\xi_1, \dots, \xi_s\} \subset N$ is an F-basis with respect to FN in the sense of Definition 4.2, then we call W a *standard basis* of N .

Remark By referring to Lemma 4.1, one may check that our definition 4.5 of a standard basis coincides with the classical Macaulay basis provided $A = K[x_1, \dots, x_n]$ is the commutative polynomial K -algebra (cf. [KR] Definition 4.2.13, Theorem 4.3.19), for, taking the \mathbb{N} -filtration FA with respect to an arbitrarily chosen positive-degree function $d(\cdot)$ on A , there are graded algebra isomorphisms $G(A) \cong A$ and $\tilde{A} \cong K[x_0, x_1, \dots, x_n]$, where $d(x_0) = 1$ and x_0 plays the role that the central regular element Z of degree 1 does in \tilde{A} . Moreover, if two-sided ideals of a weighted \mathbb{N} -filtered solvable polynomial algebra A are considered, then one may see that our definition 4.5 of a standard basis coincides with the standard basis defined in [Gol].

4.6. Definition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and N a submodule of M with the filtration $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ induced by FM . Suppose that N has a standard basis $W = \{\xi_1, \dots, \xi_m\}$ with respect to FN . If any proper subset of W cannot be a standard basis for N with respect to FN , then we call W a *minimal standard basis* of N with respect to FN .

If N is a submodule of a filtered A -module M with filtration FM , then since a standard basis of N is defined as an F-basis of N with respect to the filtration FN induced by FM , the next proposition follows from Proposition 4.4.

4.7. Proposition Let M be a filtered A -module with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$, and N a submodule of M with the induced filtration $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$. A finite subset of nonzero elements $W = \{\xi_1, \dots, \xi_s\} \subset N$ is a minimal standard basis of N with respect to FN if and only if $\sigma(W) = \{\sigma(\xi_1), \dots, \sigma(\xi_m)\}$ is a minimal homogeneous generating set of $G(N)$ if and only if $\tau(W) = \{\tilde{\xi}_1, \dots, \tilde{\xi}_m\}$ is a minimal homogeneous generating set of \tilde{N} . Hence, any two minimal standard bases of N have the same number of elements and the same number of elements of the same filtered degree.

□

Since A , $G(A)$ and \tilde{A} are all Noetherian rings, if a filtered A -module M has an F-basis V with respect to a given filtration FM , then the existence of a standard basis for a submodule N of M follows immediately from Lemma 4.1. Our next theorem shows that a standard basis for a submodule N of a filtered free A -module L can be obtained via computing a left Gröbner basis of N with respect to a graded left monomial ordering.

4.8. Theorem Let $L = \oplus_{i=1}^s Ae_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$

such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, and let $\prec_{e\text{-gr}}$ be a graded left monomial ordering on $\mathcal{B}(e)$ (see Section 3). If $\mathcal{G} = \{g_1, \dots, g_m\} \subset L$ is a left Gröbner basis for the submodule $N = \sum_{i=1}^m Ag_i$ of L with respect to $\prec_{e\text{-gr}}$, then \mathcal{G} is a standard basis for N in the sense of Definition 4.5.

Proof If $\xi \in F_q N = F_q L \cap N$ and $\xi \neq 0$, then $d_{\text{fil}}(\xi) \leq q$ and ξ has a left Gröbner representation by \mathcal{G} , that is, $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} g_j$, where $\lambda_{ij} \in K^*$, $a^{\alpha(i_j)} \in \mathcal{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, satisfying $\mathbf{LM}(a^{\alpha(i_j)} g_j) \preceq_{e\text{-gr}} \mathbf{LM}(\xi)$. Suppose $d_{\text{fil}}(g_j) = n_j$, $1 \leq j \leq m$. Since $\prec_{e\text{-gr}}$ is a graded left monomial ordering on $\mathcal{B}(e)$, by Lemma 3.1 we may assume that $\mathbf{LM}(g_j) = a^{\beta(j)} e_{t_j}$ with $\beta(j) = (\beta_{j1}, \dots, \beta_{jn}) \in \mathbb{N}^n$ and $1 \leq t_j \leq s$, such that $d(a^{\beta(j)}) + b_{t_j} = n_j$, where $d(\cdot)$ is the given positive-degree function on A . Furthermore, by the property (P2) presented in Section 1, we have

$$\mathbf{LM}(a^{\alpha(i_j)} g_j) = \mathbf{LM}(a^{\alpha(i_j)} a^{\beta(j)} e_{t_j}) = a^{\alpha(i_j) + \beta(j)} e_{t_j},$$

and it follows from Lemma 2.2, Lemma 2.7 and Lemma 3.1 that $d(a^{\alpha(i_j)}) + n_j = d(a^{\alpha(i_j)}) + d(a^{\beta(j)}) + b_{t_j} = d(a^{\alpha(i_j) + \beta(j)}) + b_{t_j} \leq q$. Hence $\xi \in \sum_{j=1}^m \left(\sum_{p_j + n_j \leq q} F_{p_j} A \right) g_j$. This shows that $F_q N = \sum_{j=1}^m \left(\sum_{p_j + n_j \leq q} F_{p_j} A \right) g_j$, i.e., \mathcal{G} is a standard basis for N .

5. Computation of Minimal F-Bases and Minimal Standard Bases

Let $A = K[a_1, \dots, a_n]$ be a weighted \mathbb{N} -filtered solvable polynomial algebra with admissible system (\mathcal{B}, \prec) and the \mathbb{N} -filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ constructed with respect to a given positive-degree function $d(\cdot)$ on A (see Section 2). In this section we show how to algorithmically compute minimal F-bases for quotient modules of a filtered free left A -module L , and how to algorithmically compute minimal standard bases for submodules of L in the case where a graded left monomial ordering $\prec_{e\text{-gr}}$ on L is employed. All notions, notations and conventions used before are maintained.

We start by a little more preparation. Let M and M' be filtered A -modules with the filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$ and $FM' = \{F_q M'\}_{q \in \mathbb{N}}$ respectively. Recall that an A -module homomorphism $\varphi: M \rightarrow M'$ is said to be a *filtered homomorphism* if $\varphi(F_q M) \subseteq F_q M'$ for all $q \in \mathbb{N}$. Let $G(A)$ be the associated \mathbb{N} -graded algebra of A and \tilde{A} the Rees algebra of A . Then naturally, a filtered homomorphism $M \xrightarrow{\varphi} M'$ induces a graded $G(A)$ -module homomorphism $G(M) \xrightarrow{G(\varphi)} G(M')$, where if $\xi \in F_q M$ and $\bar{\xi} = \xi + F_{q-1} M$ is the coset represented by ξ in $G(M)_q = F_q M / F_{q-1} M$, then $G(\varphi)(\bar{\xi}) = \varphi(\xi) + F_{q-1} M' \in G(M')_q = F_q M' / F_{q-1} M'$, and φ induces a graded \tilde{A} -module homomorphism $\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{M}'$, where if $\xi \in F_q M$ and $h_q(\xi)$ is the homogeneous element of degree q in $\tilde{M}_q = F_q M$, then $\tilde{\varphi}(h_q(\xi)) = h_q(\varphi(\xi)) \in \tilde{M}'_q = F_q M'$. Moreover, if $M \xrightarrow{\varphi} M' \xrightarrow{\psi} M''$ is a sequence of filtered homomorphisms, then $G(\psi) \circ G(\varphi) = G(\psi \circ \varphi)$ and $\tilde{\psi} \circ \tilde{\varphi} = \tilde{\psi \circ \varphi}$.

Furthermore, recall that a filtered homomorphism $M \xrightarrow{\varphi} M'$ is called a *strict filtered homomorphism* if $\varphi(F_q M) = \varphi(M) \cap F_q M'$ for all $q \in \mathbb{N}$. Note that if N is a submodule of M and $\overline{M} = M/N$, then, considering the induced filtration $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ of N and the induced filtration

$F(\overline{M}) = \{F_q \overline{M} = (F_q M + N)/N\}_{q \in \mathbb{N}}$ of \overline{M} , the inclusion map $N \hookrightarrow M$ and the canonical map $M \rightarrow \overline{M}$ are strict filtered homomorphisms. Concerning strict filtered homomorphisms and their associated graded homomorphisms, the next proposition is quoted from ([LVO], CH.I, Section 4).

5.1. Proposition Given a sequence of filtered homomorphisms

$$(*) \quad N \xrightarrow{\varphi} M \xrightarrow{\psi} M',$$

such that $\psi \circ \varphi = 0$, the following statements are equivalent.

- (i) The sequence $(*)$ is exact and φ, ψ are strict filtered homomorphisms.
- (ii) The sequence $G(N) \xrightarrow{G(\varphi)} G(M) \xrightarrow{G(\psi)} G(M')$ is exact.
- (iii) The sequence $\widetilde{N} \xrightarrow{\widetilde{\varphi}} \widetilde{M} \xrightarrow{\widetilde{\psi}} \widetilde{M}'$ is exact.

□

Let $L = \oplus_{i=1}^m A e_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq m$. Then as we have noted in Section 4, $\{e_1, \dots, e_m\}$ is an F-basis of L with respect to the good filtration FL . Let N be a submodule of L , and let the quotient module $M = L/N$ be equipped with the filtration $FM = \{F_q M = (F_q L + N)/N\}_{q \in \mathbb{N}}$ induced by FL . Without loss of generality, we assume that $\bar{e}_i \neq 0$ for $1 \leq i \leq m$, where each \bar{e}_i is the coset represented by e_i in M . Then we see that $\{\bar{e}_1, \dots, \bar{e}_m\}$ is an F-basis of M with respect to FM .

5.2. Lemma Let $M = L/N$ be as fixed above, and let $N = \sum_{j=1}^s A \xi_j$ be generated by the set of nonzero elements $U = \{\xi_1, \dots, \xi_s\}$, where $\xi_\ell = \sum_{k=1}^s f_{k\ell} e_k$ with $f_{k\ell} \in A$ and $d_{\text{fil}}(\xi_\ell) = q_\ell$, $1 \leq \ell \leq s$. The following statements hold.

- (i) If for some j , ξ_j has a nonzero term $f_{ij} e_i$ such that $d_{\text{fil}}(f_{ij} e_i) = d_{\text{fil}}(\xi_j) = q_j$ and the coefficient f_{ij} is a nonzero constant, say $f_{ij} = 1$ without loss of generality, then for each $\ell = 1, \dots, j-1, j+1, \dots, s$, the element $\xi'_\ell = \xi_\ell - f_{i\ell} \xi_j$ does not involve e_i . Putting $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$, $N' = \sum_{\xi'_\ell \in U'} A \xi'_\ell$, and considering the filtered free A -module $L' = \oplus_{k \neq i} A e_k$ with the filtration $FL' = \{F_q L'\}_{q \in \mathbb{N}}$ in which each e_k has the same filtered degree as it is in L , i.e., $d_{\text{fil}}(e_k) = b_k$, if the quotient module $M' = L'/N'$ is equipped with the filtration $FM' = \{F_q M' = (F_q L' + N')/N'\}_{q \in \mathbb{N}}$ induced by FL' , then there is a strict filtered isomorphism $\varphi: M' \cong M$, i.e., φ is an A -module isomorphism such that $\varphi(F_q M') = F_q M$ for all $q \in \mathbb{N}$.
- (ii) With the assumptions and notations as in (i), if $U = \{\xi_1, \dots, \xi_s\}$ is a standard basis of N with respect to the filtration FN induced by FL , then $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$ is a standard basis of N' with respect to the filtration FN' induced by FL' .

Proof (i) Since $f_{ij} = 1$ by the assumption, we see that every $\xi'_\ell = \xi_\ell - f_{i\ell} \xi_j$ does not involve e_i . Let $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$ and $N' = \sum_{\xi'_\ell \in U'} A \xi'_\ell$. Then $N' \subset L' = \oplus_{k \neq i} A e_k \subset L$ and $N = N' + A \xi_j$. Since $\xi_j = e_i + \sum_{k \neq i} f_{kj} e_k$, the naturally defined A -module homomorphism $M' = L'/N' \xrightarrow{\varphi} L/N = M$ with $\varphi(\bar{e}_k) = \bar{e}_k$, $k = 1, \dots, i-1, i+1, \dots, m$, is an isomorphism, where, without confusion, \bar{e}_k denotes the coset represented by e_k in M' and M respectively. It remains to

see that φ is a strict filtered isomorphism. Note that $\{e_1, \dots, e_m\}$ is an F-basis of L with respect to FL such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq m$, i.e.,

$$F_q L = \sum_{i=1}^m \left(\sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i, \quad q \in \mathbb{N},$$

that $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_m\}$ is an F-basis of L' with respect to FL' such that $d_{\text{fil}}(e_k) = b_k$, where $k \neq i$, i.e.,

$$F_q L' = \sum_{k \neq i} \left(\sum_{p_i + b_k \leq q} F_{p_i} A \right) e_k, \quad q \in \mathbb{N},$$

and that $\xi_j = e_i + \sum_{k \neq i} f_{kj} e_k$ with $q_j = d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i) = b_i$ such that $d_{\text{fil}}(f_{kj}) + b_k \leq q_j$ for all $f_{kj} \neq 0$. It follows that $\sum_{k \neq i} f_{kj} \bar{e}_k \in F_{q_j} M'$ and $\varphi(\sum_{k \neq i} f_{kj} \bar{e}_k) = \bar{e}_i \in F_{q_j} M$, thereby $\varphi(F_q M') = F_q M$ for all $q \in \mathbb{N}$, as desired.

(ii) Note that $\xi'_\ell = \xi_\ell - f_{i\ell} \xi_j$. By the assumption on ξ_j , if $f_{i\ell} \neq 0$ and $d_{\text{fil}}(f_{i\ell} e_i) = d_{\text{fil}}(\xi_\ell) = q_\ell$, then since $d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i)$ we have $d_{\text{fil}}(f_{i\ell} \xi_j) = d_{\text{fil}}(\xi_\ell) = q_\ell$. It follows that if we equip N with the filtration $FN = \{F_q N = N \cap F_q L\}_{q \in \mathbb{N}}$ induced by FL and consider the associated graded module $G(N)$ of N , then $d_{\text{gr}}(\sigma(\xi_\ell)) = d_{\text{gr}}(\sigma(f_{i\ell} \xi_j)) = d_{\text{gr}}(\sigma(f_{i\ell}) \sigma(\xi_j))$ in $G(N)$, i.e., $\sigma(\xi_\ell) - \sigma(f_{i\ell}) \sigma(\xi_j) \in G(N)_{q_\ell}$. So, if $\sigma(\xi_\ell) - \sigma(f_{i\ell}) \sigma(\xi_j) \neq 0$ then $d_{\text{fil}}(\xi'_\ell) = q_\ell$ and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell} \xi_j) = \sigma(\xi_\ell) - \sigma(f_{i\ell}) \sigma(\xi_j). \quad (1)$$

If $f_{i\ell} \neq 0$ and $d_{\text{fil}}(f_{i\ell} e_i) < d_{\text{fil}}(\xi_\ell) = q_\ell$, then $\sigma(\xi_\ell) = \sum_{d(f_{k\ell}) + b_k = q_\ell} \sigma(f_{k\ell}) \sigma(e_k)$ does not involve $\sigma(e_i)$. Also since $d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i)$, we have $d_{\text{fil}}(f_{i\ell} \xi_j) < d_{\text{fil}}(\xi_\ell) = q_\ell$. Hence $d_{\text{fil}}(\xi'_\ell) = d_{\text{fil}}(\xi_\ell) = q_\ell$ and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell} \xi_j) = \sigma(\xi_\ell). \quad (2)$$

If $f_{i\ell} = 0$, then it is clear that we have the same result as presented in (2). Now, if $U = \{\xi_1, \dots, \xi_s\}$ is a standard basis of N with respect to the induced filtration FN , then $G(N) = \sum_{\ell=1}^s G(A) \sigma(\xi_\ell)$ by Lemma 4.1. But since we have also $G(N) = \sum_{\xi'_\ell \in U'} G(A) \sigma(\xi'_\ell) + G(A) \sigma(\xi_j)$ where the $\sigma(\xi'_\ell)$ are those nonzero homogeneous elements obtained according to the above (1) and (2), it follows from Lemma 4.1 that $U' \cup \{\xi_j\}$ is a standard basis of N with respect to the induced filtration FN .

We next prove that U' is a standard basis of $N' = \sum_{\xi'_\ell \in U'} A \xi'_\ell$ with respect to the filtration $FN' = \{F_q N' = N' \cap F_q L'\}_{q \in \mathbb{N}}$ induced by FL' . Since $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_m\}$ is an F-basis of L' with respect to the filtration FL' such that each e_k has the same filtered degree as it is in L , i.e., $d_{\text{fil}}(e_k) = b_k$, it is clear that $F_q L' = L' \cap F_q L$, $q \in \mathbb{N}$, i.e., the filtration FL' is the one induced by FL . Considering the filtration FN' of N' induced by FL' , it turns out that

$$F_q N' = N' \cap F_q L' = N' \cap F_q L \subseteq N \cap F_q L = F_q N, \quad q \in \mathbb{N}. \quad (3)$$

If $\xi \in F_q N'$, then since $U' \cup \{\xi_j\}$ is a standard basis of N with respect to the induced filtration FN , the formula (3) entails that

$$\xi = \sum_{\xi'_\ell \in U'} f_\ell \xi'_\ell + f_j \xi_j \text{ with } f_\ell, f_j \in A, \quad d(f_\ell) + d_{\text{fil}}(\xi'_\ell) \leq q, \quad d(f_j) + d_{\text{fil}}(\xi_j) \leq q. \quad (4)$$

Note that every ξ'_ℓ does not involve e_i and consequently ξ does not involve e_i . Hence $f_j = 0$ in (4) by the assumption on ξ_j , and thus $\xi \in \sum_{\xi'_\ell \in U'} \left(\sum_{p_i+q_\ell \leq q} F_{p_i} A \right) \xi'_\ell$. Therefor we conclude that U' is a standard basis for N' with respect to the induced filtration FN' , as desired. \square

Combining Proposition 4.4, we now show that an analogue of ([KR], Proposition 4.7.24) and ([Li4], Proposition 4.2) for quotient modules of filtered free A -modules holds true.

5.3. Proposition Let $L = \oplus_{i=1}^m A e_i$, $M = L/N$ be as in Lemma 5.2, and suppose that $U = \{\xi_1, \dots, \xi_s\}$ is now a standard basis of N with respect to the filtration FN induced by FL . The algorithm presented below computes a subset $\{e_{i_1}, \dots, e_{i_{m'}}\} \subset \{e_1, \dots, e_m\}$ and a subset $V = \{v_1, \dots, v_t\} \subset N \cap L'$, where $m' \leq m$ and $L' = \oplus_{q=1}^{m'} A e_{i_q}$ such that

- (i) there is a strict filtered isomorphism $L'/N' = M' \cong M$, where $N' = \sum_{\ell=1}^t A v_\ell$, and M' has the filtration $FM' = \{F_q M' = (F_q L' + N')/N'\}_{q \in \mathbb{N}}$ induced by the good filtration FL' determined by the F -basis $\{e_{i_1}, \dots, e_{i_{m'}}\}$ of L' ;
- (ii) $V = \{v_1, \dots, v_t\}$ is a standard basis of $N' = \sum_{\ell=1}^t A v_\ell$ with respect to the filtration FN' induced by FL' , such that each $v_\ell = \sum_{k=1}^{m'} h_{k\ell} e_{i_k}$ has the property that $h_{k\ell} \in K^*$ implies $d_{\text{fil}}(e_{i_k}) = b_{i_k} < d_{\text{fil}}(v_\ell)$;
- (iii) $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$ is a minimal F -basis of M with respect to the filtration FM .

Algorithm 2

INPUT : $E = \{e_1, \dots, e_m\}$; $U = \{\xi_1, \dots, \xi_s\}$, where $\xi_\ell = \sum_{k=1}^m f_{k\ell} e_k$ with $f_{k\ell} \in A$,
and $d(f_{k\ell}) + b_k \leq q_\ell = d_{\text{fil}}(\xi_\ell)$, $1 \leq \ell \leq s$.

OUTPUT : $E' = \{e_{i_1}, \dots, e_{i_{m'}}\}$; $V = \{v_1, \dots, v_t\} \subset N \cap L'$, where $L' = \oplus_{k=1}^{m'} A e_{i_k}$

INITIALIZATION : $t := s$; $V := U$; $m' := m$; $E' := E$;

BEGIN

WHILE there is a $v_j = \sum_{k=1}^{m'} f_{kj} e_k \in V$ and i is the least index

such that $f_{ij} \in K^*$ with $d(f_{ij}) + b_i = d_{\text{fil}}(v_j)$ DO

set $T = \{1, \dots, j-1, j+1, \dots, t\}$ and compute

$v'_\ell = v_\ell - \frac{1}{f_{ij}} f_{i\ell} v_j$, $\ell \in T$,

$r = \#\{\ell \mid \ell \in T, v_\ell = 0\}$

$t := t - r - 1$

$V := \{v_\ell = v'_\ell \mid \ell \in T, v'_\ell \neq 0\}$

$= \{v_1, \dots, v_t\}$ (after reordered)

$m' := m' - 1$

$E' := E' - \{e_i\} = \{e_1, \dots, e_{m'}\}$ (after reordered)

END

END

Proof First note that for each $\xi_\ell \in U$, $d_{\text{fil}}(\xi_\ell)$ is determined by Lemma 2.6. Since the algorithm is clearly finite, the conclusions (i) and (ii) follow from Lemma 5.2.

To prove the conclusion (iii), by the strict filtered isomorphism $M' = L'/N' \cong M$ (or the proof of Lemma 5.2(i)) it is sufficient to show that $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$ is a minimal F-basis of M' with respect to the filtration FM' . By the conclusion (ii), $V = \{v_1, \dots, v_t\}$ is a standard basis of the submodule $N' = \sum_{\ell=1}^t Av_\ell$ of L' with respect to the filtration FN' induced by FL' such that each $v_\ell = \sum_{k=1}^{m'} h_{k\ell} e_{i_k}$ has the property that $h_{k\ell} \in K^*$ implies $d_{\text{fil}}(e_{i_k}) = b_{i_k} < d_{\text{fil}}(v_\ell)$. It follows from Lemma 4.1 that $G(N') = \sum_{k=1}^t G(A)\sigma(v_k)$ in which each $\sigma(v_k) = \sum_{d(h_{k\ell})+b_{i_k}=d_{\text{fil}}(v_k)} \sigma(h_{k\ell})\sigma(e_{i_k})$ and all the coefficients $\sigma(h_{k\ell})$ satisfy $d_{\text{gr}}(\sigma(h_{k\ell})) > 0$ (see Section 2). Noticing that $G(A)_0 = K$, $G(L') = \oplus_{k=1}^{m'} G(A)\sigma(e_{i_k})$ (Proposition 2.8) and $G(N')$ is the graded syzygy module of the graded quotient module $G(L')/G(N')$, by the classical result on finitely generated graded modules over \mathbb{N} -graded algebras with the degree-0 homogeneous part a field, we conclude that $\{\overline{\sigma(e_{i_1})}, \dots, \overline{\sigma(e_{i_{m'}})}\}$ is a minimal homogeneous generating set of $G(L')/G(N')$. On the other hand, considering the naturally formed exact sequence of strict filtered homomorphisms

$$0 \longrightarrow N' \longrightarrow L' \longrightarrow M' = L'/N' \longrightarrow 0,$$

by Proposition 5.1 we have the \mathbb{N} -graded $G(A)$ -module isomorphism $G(L')/G(N') \cong G(L'/N') = G(M')$ with $\overline{\sigma(e_{i_k})} \mapsto \sigma(\bar{e}_{i_k})$, $1 \leq k \leq m'$. Now, applying Proposition 4.4, we conclude that $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$ is a minimal F-basis of M' with respect to the filtration FM' , as desired. \square

Finally, let $L = \oplus_{i=1}^s Ae_i$ be a filtered free A -module with the filtration $FL = \{F_q L\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq s$, and let $\prec_{e\text{-gr}}$ be a graded left monomial ordering on the K -basis $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$ of L (see Section 3). Combining Theorem 3.3 and ([Li3], Theorem 3.8), the next theorem shows how to algorithmically compute a minimal standard basis.

5.4. Theorem Let $N = \sum_{i=1}^c A\theta_i$ be a submodule of L generated by the subset of nonzero elements $\Theta = \{\theta_1, \dots, \theta_c\}$, and let $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$ be the filtration of N induced by FL . Then a minimal standard basis of N with respect to FN can be obtained by implementing the following procedures:

Procudure 1. Run **Algorithm 1** presented in Section 1 with the initial input data $\Theta = \{\theta_1, \dots, \theta_c\}$ to compute a left Gröbner basis $U = \{\xi_1, \dots, \xi_m\}$ for N with respect to $\prec_{e\text{-gr}}$ on $\mathcal{B}(e)$.

Procudure 2. Let $G(N)$ be the associated graded $G(A)$ -module of N determined by the induced filtration FN . Then $G(N)$ is a graded submodule of the associated grade free $G(A)$ -module $G(L)$, and it follows from Theorem 3.3 that $\sigma(U) = \{\sigma(\xi_1), \dots, \sigma(\xi_m)\}$ is a homogeneous left Gröbner basis of $G(N)$ with respect to $\prec_{\sigma(e)\text{-gr}}$ on $\sigma(\mathcal{B}(e))$. Run **Algorithm 3** presented in ([Li3], Theorem 3.8) with the initial input data $\sigma(U)$ to compute a minimal homogeneous generating set $\{\sigma(\xi_{j_1}), \dots, \sigma(\xi_{j_t})\} \subseteq \sigma(U)$ for $G(N)$.

Procudure 3. Write down $W = \{\xi_{j_1}, \dots, \xi_{j_t}\}$. Then W is a Minimal standard basis for N by Proposition 4.7. \square

Remark By Theorem 3.5 and Proposition 4.7 it is clear that we can also obtain a minimal standard

basis of the submodule N via computing a minimal homogeneous generating set for the Rees module \tilde{N} of N , which is a graded submodule of the Rees module \tilde{L} of L . However, noticing the structure of \tilde{A} and \tilde{L} (see Theorem 2.4, Proposition 2.8) it is equally clear that the cost of working on \tilde{A} will be much higher than working on $G(N)$.

6. The Uniqueness of Minimal Filtered Free Resolutions

Let A be a weighted \mathbb{N} -filtered solvable polynomial algebra with admissible system (\mathcal{B}, \prec) and the \mathbb{N} -filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ constructed with respect to a given positive-degree function $d(\cdot)$ on A (see Section 2). In this section, by using minimal F -bases and minimal standard bases in the sense of Definition 4.3 and Definition 4.6, we define minimal filtered free resolutions for finitely generated left A -modules, and we show that such minimal resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered A -modules. All notions, notations and conventions used before are maintained.

Let $M = \sum_{i=1}^m A\xi_i$ be an arbitrary finitely generated A -module. Then, as we have noted in Section 4, M may be endowed with a good filtration $FM = \{F_q M\}_{q \in \mathbb{N}}$ with respect to an arbitrarily chosen subset $\{n_1, \dots, n_m\} \subset \mathbb{N}$, where

$$F_q M = \{0\} \text{ if } q < \min\{n_1, \dots, n_m\}; \text{ otherwise } F_q M = \sum_{i=1}^t \left(\sum_{p_i + n_i \leq q} F_{p_i} A \right) \xi_i, \quad q \in \mathbb{N}.$$

If we consider the filtered free A -module $L_0 = \oplus_{i=1}^m Ae_i$ with the good filtration $FL_0 = \{F_q L_0\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = n_i$, $1 \leq i \leq m$, then it follows from the construction of FL_0 (see Section 2) and Proposition 5.1 that the following proposition holds.

6.1. Proposition (i) There is an exact sequence of strict filtered homomorphisms

$$0 \longrightarrow N \xrightarrow{\iota} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

in which $\varphi_0(e_i) = \xi_i$, $1 \leq i \leq m$, $N = \text{Ker} \varphi_0$ with the induced filtration $FN = \{F_q N = N \cap F_q L_0\}_{q \in \mathbb{N}}$, and ι is the inclusion map. Equipping $\overline{L}_0 = L_0/N$ with the induced filtration $F\overline{L}_0 = \{F_q \overline{L}_0 = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$, it turns out that the induced A -module isomorphism $\overline{L}_0 \xrightarrow[\cong]{\overline{\varphi}_0} M$ is a strict filtered isomorphism, that is, $\overline{\varphi}_0$ satisfies $\overline{\varphi}_0(F_q \overline{L}_0) = F_q M$ for all $q \in \mathbb{N}$.

(ii) The induced sequence

$$0 \longrightarrow G(N) \xrightarrow{G(\iota)} G(L_0) \xrightarrow{G(\varphi_0)} G(M) \longrightarrow 0$$

is an exact sequence of graded $G(A)$ -modules homomorphisms, thereby $G(L_0)/G(N) \cong G(M) \cong G(L_0/N)$ as graded $G(A)$ -modules.

(iii) The induced sequence

$$0 \longrightarrow \tilde{N} \xrightarrow{\tilde{\iota}} \tilde{L}_0 \xrightarrow{\tilde{\varphi}_0} \tilde{M} \longrightarrow 0$$

is an exact sequence of graded \tilde{A} -modules homomorphisms, thereby $\tilde{L}_0/\tilde{N} \cong \tilde{M} \cong \widetilde{L_0/N}$ as graded \tilde{A} -modules.

□

Proposition 6.1(i) enables us to make the following

Convention In what follows we shall always assume that a finitely generated A -module M is of the form as presented in Proposition 5.1(i), i.e., $M = L_0/N$ and M has the good filtration $FM = \{F_q M = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$.

Comparing with the classical minimal graded free resolutions defined for finitely generated graded modules over \mathbb{N} -graded algebras with the degree-0 homogeneous part a field, the results obtained in previous sections and the preliminary we made above naturally motivate the following

6.2. Definition Let $L_0 = \bigoplus_{i=1}^m A e_i$ be a filtered free A -module with the filtration $FL_0 = \{F_q L_0\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq m$, let N be a submodule of L_0 , and let the A -module $M = L_0/N$ be equipped with the filtration $FM = \{F_q M = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$. A *minimal filtered free resolution* of M is an exact sequence of filtered A -module homomorphisms

$$\mathcal{L}_\bullet \quad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

satisfying

- (1) φ_0 is the canonical epimorphism, i.e., $\varphi_0(e_i) = \bar{e}_i$ for $e_i \in \mathcal{E}_0 = \{e_1, \dots, e_m\}$ (where each \bar{e}_i denotes the coset represented by e_i in M), such that $\varphi_0(\mathcal{E}_0)$ is a minimal F-basis of M with respect to FM (in the sense of Definition 4.3);
- (2) for $i \geq 1$, each L_i is a filtered free A -module with finite A -basis \mathcal{E}_i , and each φ_i is a strict filtered homomorphism, such that $\varphi_i(\mathcal{E}_i)$ is a minimal standard basis of $\text{Ker} \varphi_{i-1}$ with respect to the filtration induced by FL_{i-1} (in the sense of Definition 4.6).

To see that the minimal filtered free resolution introduced above is an appropriate definition for finitely generated modules over weighted \mathbb{N} -filtered solvable polynomial algebras, we now show that such a resolution is unique up to a strict filtered isomorphism of chain complexes in the category of filtered A -modules.

6.3. Theorem Let \mathcal{L}_\bullet be a minimal filtered free resolution of M as presented in Definition 6.2. The following statements hold.

- (i) The associated sequence of graded $G(A)$ -modules and graded $G(A)$ -module homomorphisms

$$G(\mathcal{L}_\bullet) \quad \cdots \xrightarrow{G(\varphi_{i+1})} G(L_i) \xrightarrow{G(\varphi_i)} \cdots \xrightarrow{G(\varphi_2)} G(L_1) \xrightarrow{G(\varphi_1)} G(L_0) \xrightarrow{G(\varphi_0)} G(M) \rightarrow 0$$

is a minimal graded free resolution of the finitely generated graded $G(A)$ -module $G(M)$.

(ii) The associated sequence of graded \tilde{A} -modules and graded \tilde{A} -module homomorphisms

$$\tilde{\mathcal{L}}_{\bullet} \quad \dots \xrightarrow{\tilde{\varphi}_{i+1}} \tilde{L}_i \xrightarrow{\tilde{\varphi}_i} \dots \xrightarrow{\tilde{\varphi}_2} \tilde{L}_1 \xrightarrow{\tilde{\varphi}_1} \tilde{L}_0 \xrightarrow{\tilde{\varphi}_0} \tilde{M} \rightarrow 0$$

is a minimal graded free resolution of the finitely generated graded \tilde{A} -module \tilde{M} .

(iii) \mathcal{L}_{\bullet} is uniquely determined by M in the sense that if M has another minimal filtered free resolution

$$\mathcal{L}'_{\bullet} \quad \dots \xrightarrow{\varphi'_{i+1}} L'_i \xrightarrow{\varphi'_i} \dots \xrightarrow{\varphi'_2} L'_1 \xrightarrow{\varphi'_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

then for each $i \geq 1$, there is a strict filtered A -module isomorphism $\chi_i: L_i \rightarrow L'_i$ such that the diagram

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_i} & L_{i-1} \\ \chi_i \downarrow \cong & & \chi_{i-1} \downarrow \cong \\ L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} \end{array}$$

is commutative, thereby $\{\chi_i \mid i \geq 1\}$ gives rise to a strict filtered isomorphism of chain complexes of filtered modules: $\mathcal{L}_{\bullet} \cong \mathcal{L}'_{\bullet}$.

Proof (i) and (ii) follow from Proposition 4.4, Proposition 4.7, and Proposition 6.1.

To prove (iii), let the sequence \mathcal{L}'_{\bullet} be as presented above. By (ii), $G(\mathcal{L}'_{\bullet})$ is another minimal graded free resolution of $G(M)$. It follows from the classical result on minimal graded free resolutions that there is a graded isomorphism of chain complexes $G(\mathcal{L}_{\bullet}) \cong G(\mathcal{L}'_{\bullet})$ in the category of graded $G(A)$ -modules, i.e., for each $i \geq 1$, there is a graded $G(A)$ -modules isomorphism $\psi_i: G(L_i) \rightarrow G(L'_i)$ such that the diagram

$$\begin{array}{ccc} G(L_i) & \xrightarrow{G(\varphi_i)} & G(L_{i-1}) \\ \psi_i \downarrow \cong & & \psi_{i-1} \downarrow \cong \\ G(L'_i) & \xrightarrow{G(\varphi'_i)} & G(L'_{i-1}) \end{array}$$

is commutative. Our aim below is to construct the desired strict filtered isomorphisms χ_i by using the graded isomorphisms ψ_i carefully. So, starting with L_0 , we assume that we have constructed the strict filtered isomorphisms $L_j \xrightarrow{\chi_j} L'_j$, such that $G(\chi_j) = \psi_j$ and $\chi_{j-1}\varphi_j = \varphi'_j\chi_j$, $1 \leq j \leq i-1$. Let $L_i = \oplus_{j=1}^{s_i} Ae_{ij}$. Since each ψ_i is a graded isomorphism, we have $\psi_i(\sigma(e_{ij})) = \sigma(\xi'_j)$ for some $\xi'_j \in L'_i$ satisfying $d_{\text{fil}}(\xi'_j) = d_{\text{gr}}(\sigma(\xi'_j)) = d_{\text{gr}}(\sigma(e_{ij})) = d_{\text{fil}}(e_{ij})$. It follows that if we construct the filtered A -module homomorphism $L_i \xrightarrow{\tau_i} L'_i$ by setting $\tau_i(e_{ij}) = \xi'_j$, $1 \leq j \leq s_i$, then $G(\tau_i) = \psi_i$. Hence, τ_i is a strict filtered isomorphism by Proposition 5.1. Since $\psi_{i-1} = G(\chi_{i-1})$, $\psi_i = G(\tau_i)$, and thus

$$\begin{aligned} \psi_{i-1}G(\varphi_i) &= G(\chi_{i-1})G(\varphi_i) = G(\chi_{i-1}\varphi_i) \\ G(\varphi'_i)\psi_i &= G(\varphi'_i)G(\tau_i) = G(\varphi'_i\tau_i), \end{aligned}$$

for each $q \in \mathbb{N}$, by the strictness of the φ_j , φ'_j , χ_j and τ_i , we have

$$\begin{aligned} G(\chi_{i-1}\varphi_i)(G(L_i)_q) &= ((\chi_{i-1}\varphi_i)(F_q L_i) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= (\chi_{i-1}(\varphi_i(L_i) \cap F_q L_{i-1}) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &\subseteq ((\chi_{i-1}\varphi_i)(L - i) \cap \chi_{i-1}(F_q L_{i-1}) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= ((\chi_{i-1}\varphi_i)(L_i) \cap F_q L'_{i-1} + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1}, \end{aligned}$$

$$\begin{aligned}
G(\varphi'_i \tau_i)(G(L_i)_q) &= ((\varphi'_i \tau_i)(F_q L_i) + F_{q-1} L'_{i-1}) / F_{q-1} L'_{i-1} \\
&= (\varphi'_i(F_q L'_i) + F_{q-1} L'_{i-1}) / F_{q-1} L'_{i-1} \\
&= (\varphi'_i(L'_i) \cap F_q L'_{i-1} + F_{q-1} L'_{i-1}) / F_{q-1} L'_{i-1} \\
&= ((\varphi'_i \tau_i)(L_i) \cap F_q L'_{i-1} + F_{q-1} L'_{i-1}) / F_{q-1} L'_{i-1}.
\end{aligned}$$

Note that by the exactness of \mathcal{L}_\bullet and \mathcal{L}'_\bullet , as well as the commutativity $\chi_{i-2}\varphi_{i-1} = \varphi'_{i-1}\chi_{i-1}$, we have $(\chi_{i-1}\varphi_i)(L_i) \subseteq \varphi'_i(L'_i) = (\varphi'_i \tau_i)(L_i)$. Considering the filtration of the submodules $(\chi_{i-1}\varphi_i)(L_i)$ and $(\varphi'_i \tau_i)(L_i)$ induced by the filtration $F L'_{i-1}$ of L'_{i-1} , the commutativity $\psi_{i-1}G(\varphi_i) = G(\varphi'_i)\psi_i$ and the formulas derived above show that both submodules have the same associated graded module, i.e., $G((\chi_{i-1}\varphi_i)(L_i)) = G((\varphi'_i \tau_i)(L_i))$. It follows from a similar proof of ([LVO], Theorem 5.7 on P.49) that

$$(\chi_{i-1}\varphi_i)(L_i) = (\varphi'_i \tau_i)(L_i). \quad (1)$$

Clearly, the equality (1) does not necessarily mean the commutativity of the diagram

$$\begin{array}{ccc}
L_i & \xrightarrow{\varphi_i} & L_{i-1} \\
\tau_i \downarrow \cong & & \chi_{i-1} \downarrow \cong \\
L'_i & \xrightarrow{\varphi'_i} & L'_{i-1}
\end{array}$$

To remedy this problem, we need to further modify the filtered isomorphism τ_i . Since $G(\chi_{i-1}\varphi_i)(\sigma(e_{i_j})) = G(\varphi'_i \tau_i)(\sigma(e_{i_j}))$, $1 \leq j \leq s_i$, if $d_{\text{fil}}(e_{i_j}) = b_j$, then by (1) and the strictness of τ_i and φ_i we have

$$\begin{aligned}
(\chi_{i-1}\varphi_i)(e_{i_j}) - (\varphi'_i \tau_i)(e_{i_j}) &\in (\varphi'_i \tau_i)(L_i) \cap F_{b_j-1} L'_{i-1} \\
&= \varphi'_i(L'_i) \cap F_{b_j-1} L'_{i-1} \\
&= \varphi'_i(F_{b_j-1} L'_i) \\
&= (\varphi'_i \tau_i)(F_{b_j-1} L_i),
\end{aligned} \quad (2)$$

and furthermore from (2) we have a $\xi_j \in F_{b_j-1} L_i$ such that $d_{\text{fil}}(e_{i_j} - \xi_j) = b_j$ and

$$(\chi_{i-1}\varphi_i)(e_{i_j}) = (\varphi'_i \tau_i)(e_{i_j} - \xi_j), \quad 1 \leq j \leq s_i. \quad (3)$$

Now, if we construct the filtered homomorphism $L_i \xrightarrow{\chi_i} L'_i$ by setting $\chi_i(e_{i_j}) = \tau_i(e_{i_j} - \xi_j)$, $1 \leq j \leq s_i$, then since $\tau_i(\xi_j) \in F_{b_j-1} L'_i$, it turns out that

$$G(\chi_i)(\sigma(e_{i_j})) = G(\tau_i)(\sigma(e_{i_j} - \xi_j)) = G(\tau_i)(\sigma(e_{i_j})) = \psi_i(\sigma(e_{i_j})), \quad 1 \leq j \leq s_i,$$

thereby $G(\chi_i) = \psi_i$. Hence, χ_i is a strict filtered isomorphism by Proposition 5.1, and moreover, it follows from (3) that we have reached the following diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\varphi_{i+1}} & L_i & \xrightarrow{\varphi_i} & L_{i-1} & \xrightarrow{\varphi_{i-1}} & \cdots \\
& & \chi_i \downarrow \cong & & \chi_{i-1} \downarrow \cong & & \\
\cdots & \xrightarrow{\varphi'_{i+1}} & L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} & \xrightarrow{\varphi'_{i-1}} & \cdots
\end{array}$$

in which $\chi_{i-1}\varphi_i = \varphi'_i \chi_i$. Repeating the same process to getting the desired χ_{i+1} and so on, the proof is thus finished.

7. Computation of Minimal Finite Filtered Free Resolutions

Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathcal{B}, \prec_{gr})$ in which \prec_{gr} is a graded monomial ordering with respect to some given positive-degree function $d(\cdot)$ on A (see Section 2). Thereby A is turned into a weighted \mathbb{N} -filtered solvable polynomial algebra with the filtration $FA = \{F_p A\}_{p \in \mathbb{N}}$ constructed with respect to the same $d(\cdot)$ (see Example (2) of Section 2). Our aim in this section is to give algorithmic procedures for computing minimal finite filtered free resolutions over A (in the sense of Definition 6.2). All notions, notations and conventions used before are maintained.

We start by an arbitrary free left A -module $L = \oplus_{i=1}^m A e_i$ with a left monomial ordering \prec_e on the K -basis $\mathcal{B}(e)$ of L . As in Section 1 we write $S_\ell(\xi_i, \xi_j)$ for the left S-polynomial of two elements $\xi_i, \xi_j \in L$. Let $N = \sum_{i=1}^m A \xi_i$ be a submodule of L generated by the set of nonzero elements $U = \{\xi_1, \dots, \xi_m\}$. We first demonstrate how to calculate a generating set of the syzygy module $\text{Syz}(U)$ by means of a left Gröbner basis of N . To this end, let $\mathcal{G} = \{g_1, \dots, g_t\}$ be a left Gröbner basis of N produced by running **Algorithm 1** (presented in Section 1) with the initial input data U and the ordering \prec_e , then every nonzero left S-polynomial $S_\ell(g_i, g_j)$ has a left Gröbner representation $S_\ell(g_i, g_j) = \sum_{i=1}^t f_i g_i$ with $\mathbf{LM}(f_i g_i) \preceq_e \mathbf{LM}(S_\ell(g_i, g_j))$ whenever $f_i \neq 0$ (note that such a representation is obtained by using the division by \mathcal{G} during executing the WHILE loop in **Algorithm 1**). Considering the syzygy module $\text{Szy}(\mathcal{G})$ of \mathcal{G} in the free A -module $L_1 = \oplus_{i=1}^t A \varepsilon_i$, if we put

$$s_{ij} = f_1 \varepsilon_1 + \dots + \left(f_i - \frac{a^{\gamma-\alpha(i)}}{\mathbf{LC}(a^{\gamma-\alpha(i)} \xi_i)} \right) \varepsilon_i + \dots + \left(f_j + \frac{a^{\gamma-\alpha(j)}}{\mathbf{LC}(a^{\gamma-\alpha(j)} \xi_j)} \right) \varepsilon_j + \dots + f_t \varepsilon_t,$$

$\mathcal{S} = \{s_{ij} \mid 1 \leq i < j \leq t\}$, then it can be shown, actually as in the commutative case (cf. [AL], Theorem 3.7.3), that \mathcal{S} generates $\text{Szy}(\mathcal{G})$. However, by employing an analogue of the Schreyer ordering $\prec_{s-\varepsilon}$ on the K -basis $\mathcal{B}(\varepsilon) = \{a^\alpha \varepsilon_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq m\}$ of L_1 induced by \mathcal{G} with respect to \prec_e , which is defined subject to the rule: for $a^\alpha \varepsilon_i, a^\beta \varepsilon_j \in \mathcal{B}(\varepsilon)$,

$$a^\alpha \varepsilon_i \prec_{s-\varepsilon} a^\beta \varepsilon_j \Leftrightarrow \begin{cases} \mathbf{LM}(a^\alpha g_i) \prec_e \mathbf{LM}(a^\beta g_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha g_i) = \mathbf{LM}(a^\beta g_j) \text{ and } i < j, \end{cases}$$

there is indeed a much stronger result, namely the noncommutative analogue of Schreyer's Theorem [Sch] (cf. Theorem 3.7.13 in [AL] for free modules over commutative polynomial algebras; Theorem 4.8 in [Lev] for free modules over solvable polynomial algebras):

7.1. Theorem With respect to the left monomial ordering $\prec_{s-\varepsilon}$ on $\mathcal{B}(\varepsilon)$ as defined above, the following statements hold.

(i) Let s_{ij} be determined by $S_\ell(g_i, g_j)$, where $i < j$, $\mathbf{LM}(g_i) = a^{\alpha(i)} e_s$ with $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$, and $\mathbf{LM}(g_j) = a^{\alpha(j)} e_s$ with $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$. Then $\mathbf{LM}(s_{ij}) = a^{\gamma-\alpha(j)} \varepsilon_j$, where $\gamma = (\gamma_1, \dots, \gamma_n)$ with each $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$.

(ii) \mathcal{S} is a left Gröbner basis of $\text{Syz}(\mathcal{G})$, thereby \mathcal{S} generates $\text{Syz}(\mathcal{G})$.

□

To go further, again let $\mathcal{G} = \{g_1, \dots, g_t\}$ be the left Gröbner basis of N produced by running **Algorithm 1** with the initial input data $U = \{\xi_1, \dots, \xi_m\}$. Using the usual matrix notation for convenience, we have

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = U_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

where the $m \times t$ matrix $U_{m \times t}$ (with entries in A) is obtained by the division by \mathcal{G} , and the $t \times m$ matrix $V_{t \times m}$ (with entries in A) is obtained by keeping track of the reductions during executing the WHILE loop of **Algorithm 1**. By Theorem 7.1, we may write $\text{Syz}(\mathcal{G}) = \sum_{i=1}^r A\mathcal{S}_i$ with $\mathcal{S}_1, \dots, \mathcal{S}_r \in L_1 = \oplus_{i=1}^t A\varepsilon_i$; and if $\mathcal{S}_i = \sum_{j=1}^t f_{ij}\varepsilon_j$, then we write \mathcal{S}_i as a $1 \times t$ row matrix, i.e., $\mathcal{S}_i = (f_{i1} \dots f_{it})$, whenever matrix notation is convenient in the according discussion. At this point, we note also that all the \mathcal{S}_i may be written down during executing the WHILE loop of **Algorithm 1** successively. Furthermore, we write $D_{(1)}, \dots, D_{(m)}$ for the rows of the matrix $D_{m \times m} = U_{m \times t}V_{t \times m} - E_{m \times m}$ where $E_{m \times m}$ is the $m \times m$ identity matrix. The following proposition is a noncommutative analogue of ([AL], Theorem 3.7.6).

Proposition 7.2. With notation fixed above, the syzygy module $\text{Syz}(U)$ of $U = \{\xi_1, \dots, \xi_m\}$ is generated by

$$\{\mathcal{S}_1 V_{t \times m}, \dots, \mathcal{S}_r V_{t \times m}, D_{(1)}, \dots, D_{(m)}\},$$

where each $1 \times m$ row matrix represents an element of the free A -module $\oplus_{i=1}^m A\omega_i$.

Proof Since

$$0 = \mathcal{S}_i \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i1} \dots f_{it}) \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i1} \dots f_{it}) V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

we have $\mathcal{S}_i V_{t \times m} \in \text{Syz}(U)$, $1 \leq i \leq r$. Moreover, since

$$\begin{aligned} D_{m \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} &= (U_{m \times t} V_{t \times m} - E_{m \times m}) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\ &= U_{m \times t} V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\ &= U_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = 0, \end{aligned}$$

we have $D_{(1)}, \dots, D_{(r)} \in \text{Syz}(U)$.

On the other hand, if $H = (h_1 \dots h_m)$ represents the element $\sum_{i=1}^m h_i \omega_i \in \oplus_{i=1}^m A \omega_i$ such that $H \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = 0$, then $0 = H U_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}$. This means $H U_{m \times t} \in \text{Syz}(\mathcal{G})$. Hence, $H U_{m \times t} = \sum_{i=1}^r f_i \mathcal{S}_i$ with $f_i \in A$, and it follows that $H U_{m \times t} V_{t \times m} = \sum_{i=1}^r f_i \mathcal{S}_i V_{t \times m}$. Therefore,

$$\begin{aligned} H &= H + H U_{m \times t} V_{t \times m} - H U_{m \times t} V_{t \times m} \\ &= H(E_m - U_{m \times t} V_{t \times m}) + \sum_{i=1}^r f_i \mathcal{S}_i V_{t \times m} \\ &= -H D_{m \times m} + \sum_{i=1}^r f_i (\mathcal{S}_i V_{t \times m}). \end{aligned}$$

This shows that every element of $\text{Syz}(U)$ is generated by $\{\mathcal{S}_1 V_{t \times m}, \dots, \mathcal{S}_r V_{t \times m}, D_{(1)}, \dots, D_{(m)}\}$, as desired. \square

Next, we recall the noncommutative version of Hilbert's syzygy theorem for solvable polynomial algebras. For a constructive proof of Hilbert's syzygy theorem by means of Gröbner bases respectively in the commutative case and the noncommutative case, we refer to (Corollary 15.11 in [Eis]) and (Section 4.4 in [Lev]).

7.3. Theorem Let $A = K[a_1, \dots, a_n]$ be an arbitrary solvable polynomial algebra with admissible system (\mathcal{B}, \prec) . Then every finitely generated (left) A -module M has a free resolution

$$0 \longrightarrow L_s \longrightarrow L_{s-1} \longrightarrow \dots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

where each L_i is a free A -module of finite rank and $s \leq n$. It follows that M has projective dimension $\text{p.dim}_A M \leq s$, and that A has global homological dimension $\text{gl.dim} A \leq n$. \square

Now, we are ready to reach the goal of this section.

7.4. Theorem Let $A = K[a_1, \dots, a_n]$ be the solvable polynomial algebra fixed in the beginning of this section, and let $L_0 = \oplus_{i=1}^m Ae_i$ be a filtered free A -module with the filtration $FL_0 = \{F_q L_0\}_{q \in \mathbb{N}}$ such that $d_{\text{fil}}(e_i) = b_i$, $1 \leq i \leq m$. If $N = \sum_{i=1}^s A\xi_i$ is a finitely generated submodule of L_0 and the quotient module $M = L_0/N$ is equipped with the filtration $FM = \{F_q M = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$. Then M has a minimal filtered free resolution of length $d \leq n$ (in the sense of Definition 6.2):

$$\mathcal{L}_\bullet \quad 0 \longrightarrow L_d \xrightarrow{\varphi_d} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which can be constructed by implementing the following procedures:

Procedure 1. Fix a graded left monomial ordering $\prec_{e\text{-}gr}$ on the K -basis $\mathcal{B}(e)$ of L_0 (see Section 3), and run **Algorithm 1** with the initial input data $U = \{\xi_1, \dots, \xi_s\}$ to compute a left Gröbner basis $\mathcal{G} = \{g_1, \dots, g_z\}$ for N , so that N has the standard basis \mathcal{G} with respect to the induced filtration $FN = \{F_q N = N \cap F_q L_0\}_{q \in \mathbb{N}}$ (Theorem 4.8).

Procedure 2. Run **Algorithm 2** (presented in Section 5) with the initial input data $E = \{e_1, \dots, e_m\}$ and $\mathcal{G} = \{g_1, \dots, g_z\}$ to compute a subset $\mathcal{E}'_0 = \{e_{i_1}, \dots, e_{i_{m'}}\} \subset \mathcal{E}_0 = \{e_1, \dots, e_m\}$ and a subset $V = \{v_1, \dots, v_t\} \subset N \cap L'_0$ such that there is a strict filtered isomorphism $L'_0/N' = M' \cong M$, where $L'_0 = \oplus_{q=1}^{m'} Ae_{i_q}$ with $m' \leq m$ and $N' = \sum_{k=1}^t Av_k$, and such that $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$ is a minimal F-basis of M with respect to the filtration FM .

For convenience, after accomplishing Procedure 2 we may assume that $\mathcal{E}_0 = \mathcal{E}'_0$, $U = V$ and $N = N'$. Accordingly we have the short exact sequence $0 \longrightarrow N \longrightarrow L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$ such that $\varphi_0(\mathcal{E}_0) = \{\bar{e}_1, \dots, \bar{e}_m\}$ is a minimal F-basis of M with respect to the filtration FM .

Procedure 3. With the initial input data $U = V$, implements the procedures presented in Theorem 5.4 to compute a minimal standard basis $W = \{\xi_{j_1}, \dots, \xi_{j_{m_1}}\}$ for N with respect to the induced filtration FN .

Procedure 4. Computes a generating set $U_1 = \{\eta_1, \dots, \eta_{s_1}\}$ of $N_1 = \text{Syz}(W)$ in the free A -module $L_1 = \oplus_{i=1}^{m_1} A\varepsilon_i$ by running **Algorithm 1** with the initial input data W and using Proposition 7.2.

Procedure 5. Construct the strict filtered exact sequence

$$0 \longrightarrow N_1 \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where the filtration FL_1 of L_1 is constructed by setting $d_{\text{fil}}(\varepsilon_k) = d_{\text{fil}}(\xi_{j_k})$, $1 \leq k \leq m_1$, and φ_1 is defined by setting $\varphi_1(\varepsilon_k) = \xi_{j_k}$, $1 \leq k \leq m_1$. If $N_1 \neq 0$, then, with the initial input data $U = U_1$, repeat Procedure 3 – Procedure 5 for N_1 and so on.

By Theorem 6.3, a minimal filtered free resolution \mathcal{L}_\bullet of M gives rise to a minimal graded free resolution $G(\mathcal{L}_\bullet)$ of $G(M)$. Since $G(A) = K[\sigma(a_1), \dots, \sigma(a_n)]$ is a solvable polynomial algebra by Theorem 2.4, it follows from Theorem 7.3 that $G(\mathcal{L}_\bullet)$ terminates at a certain step, i.e., $\text{Ker}G(\varphi_d) = 0$ for some d . But $\text{Ker}G(\varphi_d) = G(\text{Ker}\varphi_d)$ by Proposition 5.1, where $\text{Ker}\varphi_d$ has the filtration induced by FL_d , thereby $G(\text{Ker}\varphi_d) = 0$. Consequently $\text{Ker}\varphi_d = 0$ by classical filtered modules theory, thereby a minimal finite filtered free resolution of length $d \leq n$ is achieved for M .

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